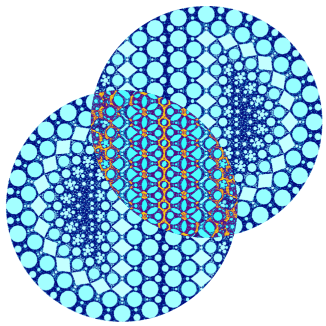
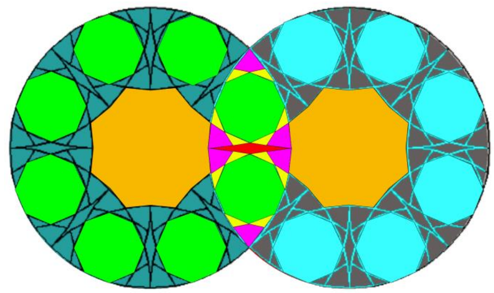
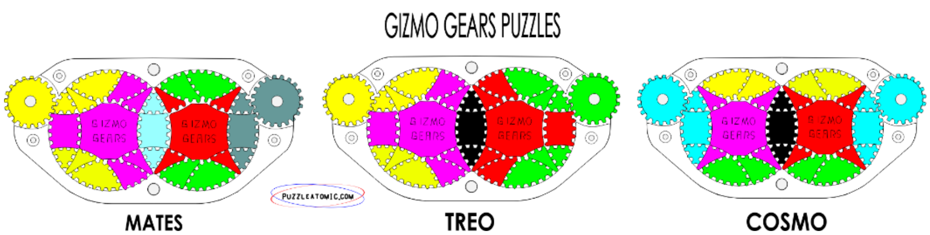
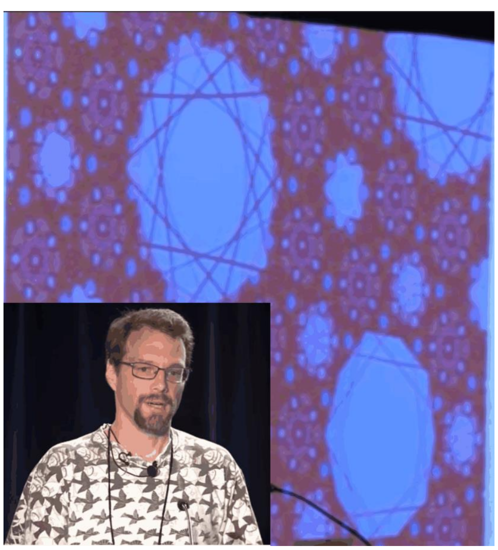
Circle Puzzler's Manual Part 2

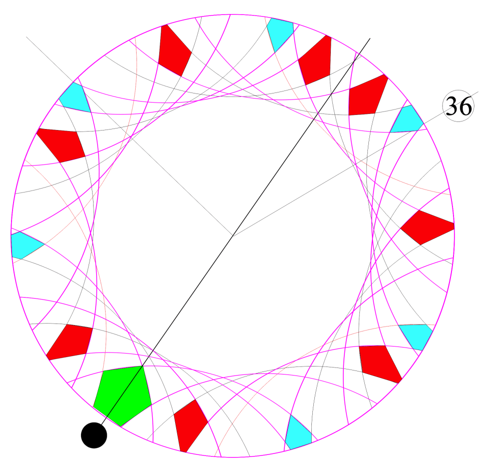
By Doug Engel











Bilateral Symmetry Clock, t=36. Space must expand as the cuts increase to keep piece size standard. New pentagonal symmetric particles are created as cuts increase.

Front page

Circle Puzzler's Manual Part 2

By Doug Engel 2019

Being an update to Circle Puzzler's Manual, ©1986

https://www.jaapsch.net/puzzles/circleman.htm

https://www.puzzleatomic.com/circleman.htm

Printed and online in United States of America

https://www.puzzleatomic.com/circlemanpart2.pdf

Available free online but hard copies must be purchased.

Email dengel99@aol.com for hard copy availability

First printing rev. zero Dec. 2019

 Links are not guaranteed to remain active. We will attempt to note or remove inactive links in updated revisions or releases.

Dedicated to my wife Neva and all the puzzle designers, solvers and collectors.

Special thanks to Jaap Scherphuis who originally put CPM online making it widely available.

Thanks to Bob Hearn for his super code, producing the first circle puzzle fractals and for his numerous talks clarifying Rc (critical radius) and how these fractals are produced.

Thanks to Jason Smith for his Gizmo Gears Deluxe Render showing fractal movies and his fractal research.

Thanks to Brandon Enright for his fractal code and his extensive mathematical analysis of Rc.

Thanks to Andreas Nortmann for putting the Gizmo Gears jumbling question on the forum after Battle Gears failed and I suggested Gizmo Gears might be worth a try.

Thanks to Oskar van Deventer's energetic input and puzzle innovations and alerting me.

Thanks to Bram Cohen for first suggesting the investigations into jumbling puzzles.

Thanks to Carl Hoff for many enlightening comments.

Thanks to Mathologer, Burkard Polster for his you tube's and to Roice Nelson for his Magictile.

Thanks to many others on the Twisty Puzzle Forum for the lively investigation, over 450 posts, that produced many insights into these fractals.

Including: David Litwin, Guilty Bystander(Landon Kryger), Jared, Kattenvriendin, Rubikcollector123, Will_57, VeryWetPaint, Eric, KelvinS …

Thanks to Martin Gardner (Hexaflexagons), Tom Rodgers (G4G), Jerry Slocum (IPP), Tom Cutrofello, Nick Baxter, Scott Morris (Omni), Erno Rubik, Doug Hofstadter, …

Acknowledgements Twisty Puzzle Forum …, Wikipedia, …

Circle Puzzler's Manual Part 2

(A tale of circle puzzle jumbling)

Definition of some of the words used with circle puzzle fractals (read only if you are unfamiliar with any of these terms). You can return here as needed.

Fractal: A self-similar figure with fractional dimension. For instance, the world famous Mandelbrot set is a fractal. You can zoom in on various parts of it and each zoom looks fairly similar to the previous one.

Doctrinaire: A twisty 3D Rubik type puzzle or flat circle type puzzle where each move of one group of pieces allows you to the chose the next possible move of a group of pieces without restriction.

Bandaged: A twisty puzzle where at least one pair of adjacent pieces are attached or glued together. In order to move when the glue would prevent a move you want to make you must cut the two pieces apart, then make your move.

Jumble or jumbling: This is the key concept that leads to the Gizmo/Hearn/circle puzzle fractals. It happens when you find a puzzle where every time you cut through a bandaged area you end up cutting as piece of it again until an infinity of pieces are the result. This in turn can produce amazing and beautiful fractal graphics and crystallographic effects, depending on the rotation generators used by the computer program (code) and various other parameters used to generate the images such as how points are colored and so forth.

Critical Radius R, Rc, Rcx: The critical radius happens when the radius of the two circles becomes large enough to cause jumbling. It has been proven in the forum, to stay critical from then on as the radius grows larger all the way to infinity. I have added Rc as a parameter for the critical radius and Rcx >= Rc, where Rcx can be Rc or any radius larger than Rc.

Surd: Used in this manual to mean irrational.

N: N is the smallest possible rotation you can make for a group of pieces. For instance if N is 4 then the smallest rotation is 360/4 = 90 degrees.

Gizmo Gears: The two circle puzzle showing both 3 way and 4 way symmetry. The question was posed "Are the Gizmo Gears Jumbling" on the forum: http://twistypuzzles.com/forum/viewtopic.php?f=1&t=25752&sid=94f858d2f15227ed1a3b97e453f75f20

Table of Contents

Some of the Motivation for this Update

CPM classifies various kinds of circle puzzles. It did not classify all possible types that can be designed (unlimited!) but did suggest various sets that some elementary designs are elements of. CPM is a way by which one can specify the major geometry aspects of some given circle puzzle. There are some examples of this in CPM. One feature of the manual was the idea that symmetrical deep cut (two or more pieces in the circle intersections) circle puzzles can generate an infinite number of pieces if the intersecting circle radii are increased without limit (increased to infinity). For instance, an illustration of hexagonally symmetrical circle puzzles, in CPM, with radii increase, show rapid increase in the number and shapes of the pieces as the two circles approach each other (R=radius of the two circles increases). The surprising thing is that the number of different congruent pieces increases very slowly. At infinite radii it was called an infinite or perfect black circle in the CPM: https://www.puzzleatomic.com/images/circleman/p35.gif. Since then at least one designer, Oskar van Deventer has produced some designs not classified in CPM.

Any errors, technically, editorially, graphically, network links, and/or in mentioning other's efforts and opinions, the fault is mine. I will try to make corrections in a revised edition. For instance, my calling N=infinity might be objected to. The Gizmo gears is designed to have two N's, N=3, and N=4 or N=(3,4), (can Gizmo Gears (3,4) restriction be turned into a fractal?). Gizmo has both triangular and square symmetry in a bandaged mode. Gizmo with unbandaging attempted is 3x4 or N=12. Thus, saying N=infinity if theta (here theta=minimum turn angle) is irrational seems correct. It drives home the point that if N never closes on itself it must become very small. Theta irrational makes it possible to have single circle fractals, or (pseudo fractals?) in an infinity of different ways which is explored below. When an infinity of cuts has been made based on irrational theta, then N can be turned an infinitesimal amount before the next possible turn. Bob Hearn mentions N being irrational on the Gizmo forum "… of course we'll always get jumbling when N is irrational." In this case he means the same thing, that the rotation angle, is based on an irrational number.

This brings up another controversy that was on the forum, "can any of these really be considered fractals?". I believe they are all fractal-like, even the N=2,3,4,6 ones with no critical radius, except infinity, because at a large radius the pieces become increasingly numerous and complex and follow repeating patterns, in a time based or progression based manner. To see the pieces, you would need to use magnification of an infinitesimal but finite(!) area, for the N=2,3,4,6 ones when R ~ infinity. Infinity is a paradoxical concept when you try to tame it. If you are not familiar with N, R, jumble, etc. read on or refer to the word definitions area. Jumble means a puzzle that cannot be completely 'unbandaged' by cutting thru the 'bandages'. Bandage means two pieces 'glued' together so you are prevented from making a move until you cut thru the piece, creating two pieces. You cut thru 'bandaged' pieces to try to make it so that a move is always possible and therefore unbandage it. If, after continuing to make cuts, it is not unbandaged then it continues to forever be cut into more pieces, and it is said to jumble. The word jumble and jumbling was first introduced by Adam Cowan who designed the Bevel cube, here is a link by Bram and Oskar: http://twistypuzzles.com/articles/other-twistypuzzlesthatjumble/ Bram probably introduced 'bandaged' and 'bandaging' and perhaps 'doctrinaire' for non-bandaged puzzles. I may sometimes use the word tame for doctrinaire. Bram, after many discussions with twisty puzzlers provided a mathematical definition of jumbling, when the pieces turn to Bram's dust. Here is what David Littwin has to say "…As for history, the feature was first described by Okamoto when noting the motion of the helicopter cube: https://www.twistypuzzles.com/forum/viewtopic.php?p=48762#p48762 He and Adam Cowan both built this puzzle independently about the same time and Okamoto wasn't sure if Adam's jumbled. He uses the term "transformed". Later, in the same thread, Adam Cowan refers to "jumbled", I believe for the first time: https://www.twistypuzzles.com/forum/viewtopic.php?p=48777#p48777 Shortly thereafter Bram dives in on the matter but I think Adam coined it." I still give Bram great credit for pushing this forward. DE

The CPM was well received among puzzlers interested in twisty puzzles such as Jaap Scherphuis, and others. I was invited to a meeting in San Francisco several years ago (2010, I think) which I attended with Bram Cohen, Oskar Van Deventer, David Litwin, Tom Rodgers, Tom Cutrofello, my wife Neva, daughter Gayle and several others. Bram, David, Oskar and others spoke about various ideas concerning the design of twisty puzzles and where it might lead. I have forgotten who organized the meeting.

One interesting subject at the meeting was circle puzzles, and I listened intently as Bram and others talked about the idea of a twisty puzzle turning into dust as you attempt to make it symmetrical at some R. It was Bram Cohen that first suggested that we should research this idea to find out if it was true and what happens when a puzzle begins to turn to dust. Also, what is the simplest puzzle that produces the dust phenomenon.

A few years later I got an email from Andreas Nortmann, administrator of the twisty puzzle museum. He said an attempt to turn the Battle Gears puzzle into dust was not considered successful since it has an unbandaging at a smaller radius. It was the first I had ever heard that anyone was working on this and writing programs to see what happens. My immediate thought was to suggest to Andreas that they should try the “Gizmo Gears”, GM, puzzle design, which he agreed to do. GM has both triangular and square symmetry. This means its true symmetry is 3x4=12 or dodecagonal.

After many amazing productions of fractals by Bob Hearn and others on the forum, the Gizmo Gears produced the ‘dust’ and many beautiful fractals. But since it is based on a dodecagon it is not the simplest circle puzzle to produce these fractals. That honor goes to a two-circle puzzle having pentagonal symmetry. At that time no simple physical puzzle existed with pentagonal symmetry that I know of, but I did mention possible Penrose circle puzzles in CPM. However, Bob Hearn’s code could be used to investigate different symmetries and when 5 was plugged into his code it soon produced many amazing fractals. Bob Hearn noticed that N=5 fractals graphically produce the Penrose non periodic tiling’s discovered by the British polymath Sir Roger Penrose (in a kind of dust pattern + fractal form). I have always been a fan of this tiling since it first appeared in Martin Gardner’s Mathematical Games column. This brings up a strange (but not important) question: ‘What location does the circle puzzle Penrose fractal tiling assume in the infinite non periodic tiling?’. Now it is revealed as a circle puzzle fractal, a **Hearn Fractal**. From this I conclude that the

Penrose tiling itself is a special kind of crystal 'cfractal' since it is non periodic but self-similar across the infinite plane.

After presenting some of my own research I will present some of the latest work being done by Oskar and others with circle puzzles. Some of my circle puzzles I have sold to solvers and collectors will be pictured. Some of Bob's fractal graphics no longer appear on the forum so if they are not presented here that is why, unless I can get access to them before this is finished. Some of this material was obtained from the Twisty Puzzle Forum about the Gizmo Gears subject. The Forum is a well maintained and lively site that covers the constantly expanding field of twisty puzzles.
http://twistypuzzles.com/forum/viewtopic.php?f=1&t=25752&hilit=gizmo&start=50

Introduction to some Gizmo Fractals

Here is Figure 1 showing an unbandaging of Battle Gears. This was published on the Twisty Puzzle Forum by 'Guilty Bystander'. I have taken that image and colored it by overlaying polygons. These polygons probably tile the plane in only one way. This unbandaging was done at a slightly larger radius than the Battle Gears puzzle therefore disqualifying Battle Gears as the first and simplest

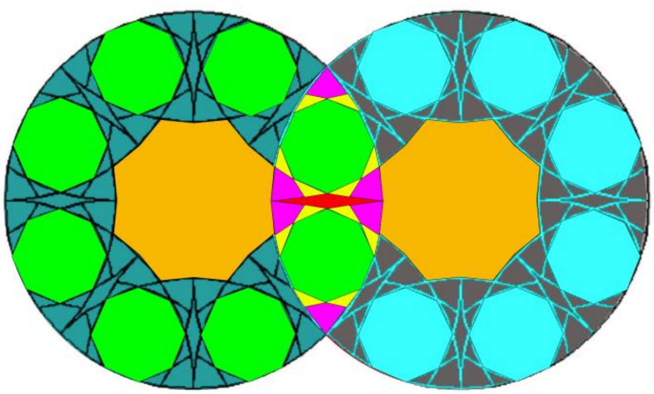
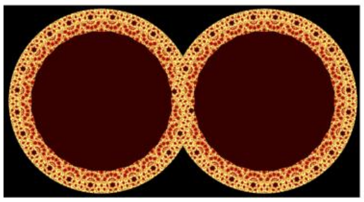


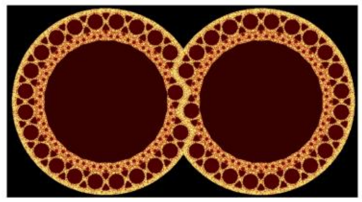
Figure 1

physical jumbling puzzle.

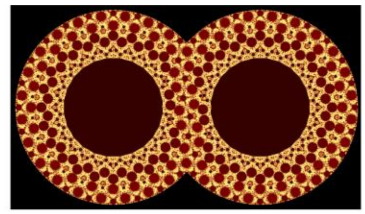
Figure 1.1 shows Jason Smith's Gizmo Render for N=24 with 9 different R values. The screen shots were captured from the render published on the deluxerender.com website. N is large so R goes critical almost immediately. Watching it in action, sudden large changes in appearance pop up even though R increases minutely. Watching this you see that themes tend to repeat themselves and grow to a maximum then suddenly undergo a major change. My favorite is the last one with R=2.91, almost where the render stops. It looks like a 3D cylinder is pushing into the centers of the two disks. The fractals are mysterious in being so varied yet having a precise crystal-like appearance. It would be interesting to see what happens as R grows a lot larger. As fractals there is a lot to digest here. As art Escher and Durer would love this stuff. As mathematics there are aspects of this will become Phd's. But seeing shot nine here does my heart a world of good. Perhaps my life of errors has not been totally wasted.



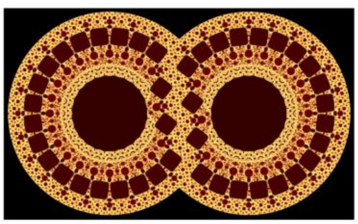
N24, R1.13999



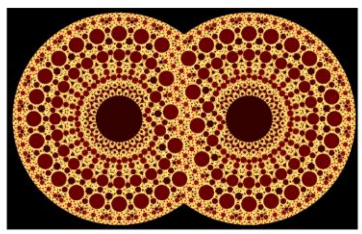
N24, R1.20999



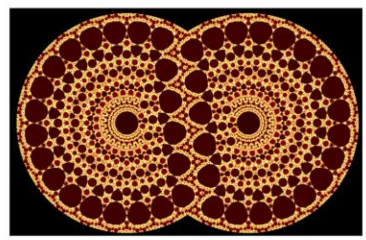
N24, R1.32999



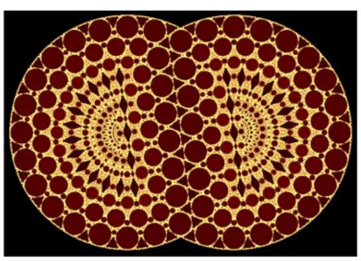
N24, R1.5999



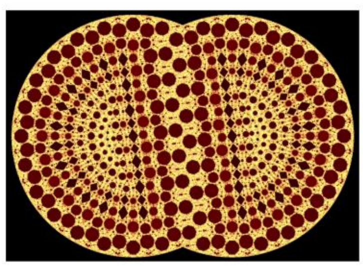
N24, R1.636



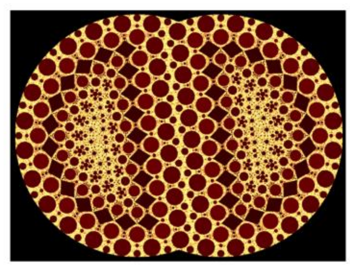
N24, R1.809



N24, R2.000



N24, R2.1500



N24, R2.91

Deluxe Render.com showing Gizmo Render for N24, R2.91

Figure 1.1

Figure 1.2, right, is screen shot 9 above modified into a helical torus. It shows the 3D look of this shot.

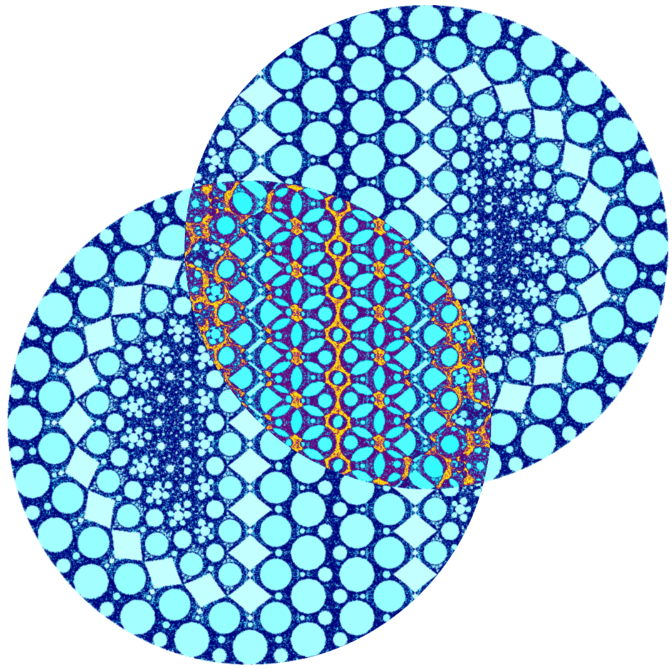
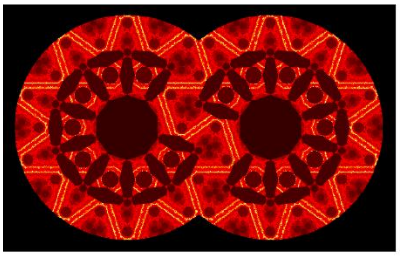


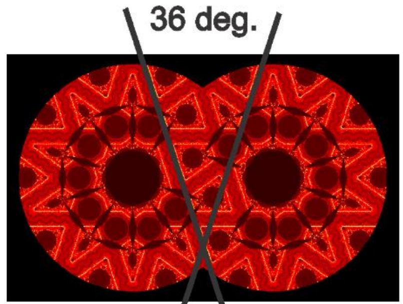
Figure 1.2

Discussion and analysis of some fractal screen shots.

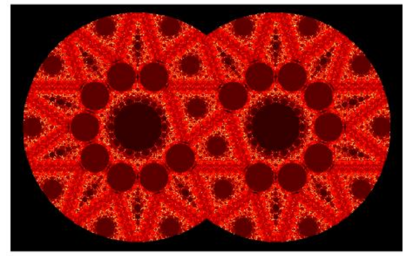
Figure 2 below shows six screen shots of Jason Smiths circle puzzle fractal code which is published on the deluxerender.com website. I have not been able to contact Jason Smith but I assume he and Bob are OK with me using these screenshots since it was my suggestion to use Gizmo to try jumbling. Gizmo was deemed to be jumbling since it is right at the critical radius. Figure 1 illustrates the wide variety of pseudo crystallographic patterns possible with just one N. Note the asymmetry of the patterns along with the symmetry. All of them appear to have 180 degree rotation symmetry. The asymmetry preserves a 36-degree tilt = 360/10 degrees in each of the patterns once R becomes equal to or greater than the critical radius. Call Rc = critical radius and let Rcx be >= Rc (where Rc means becomes critical). These fractals are multitudinous infinities. Between any two rational Rcx there are an infinity of irrational Rcx. This means that between any two rational Rcx there are an infinity of different fractals for a given N and just two circles. In fact, there are an infinity of fractals between any two Real Rcx. If the two Rcx are very close together the question arises can there be a big change in appearance? This might be true or close to true since you often see big, sudden changes when watching this Jason's Gizmo Render movie.



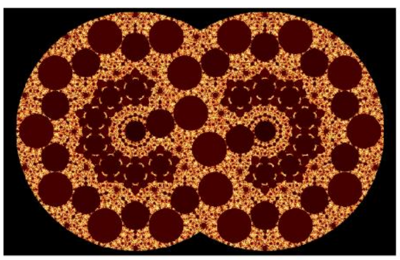
N=10, R=1.59



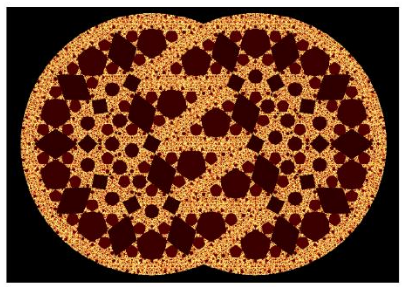
N=10, R=1.62



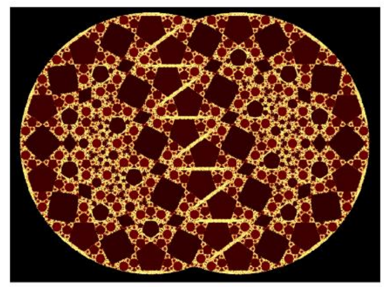
N=10, R=1.66



N=10, R=1.84



N=10, R=2.52



N=10, R=2.98

Six screen saves of Jason Smith's Jumbler code for N=10 showing the wide variation in the the fractal patterns as R increases. There are an infinity of fractals between two real numbers of Rcx, (Rcx>=critical).

Figure 2

Taking the upper left screen shots and rotating the two circles you can examine the symmetry/asymmetry of these fractals. This is a way to increase appreciation of these fractals. Figure 2.1 below shows this rotating process for five rotations.

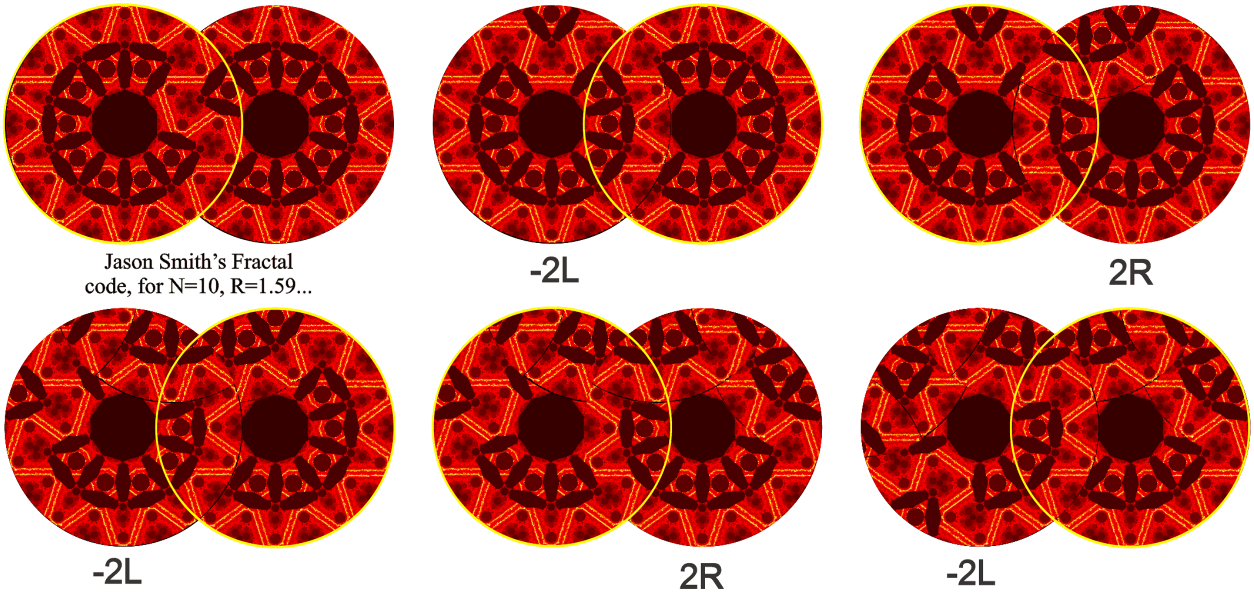


Figure 2.1

An attempt to make the pattern symmetric is shown if Figure 3.1 below. This seems to come close but does not quite succeed.

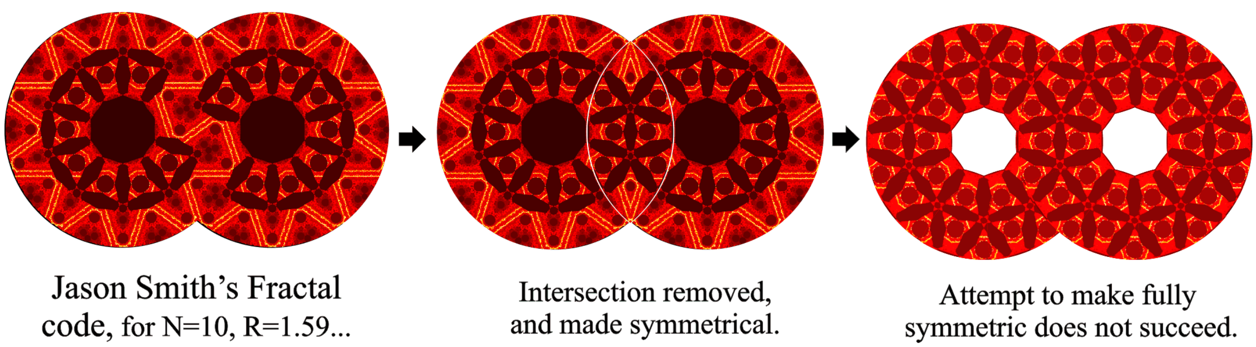
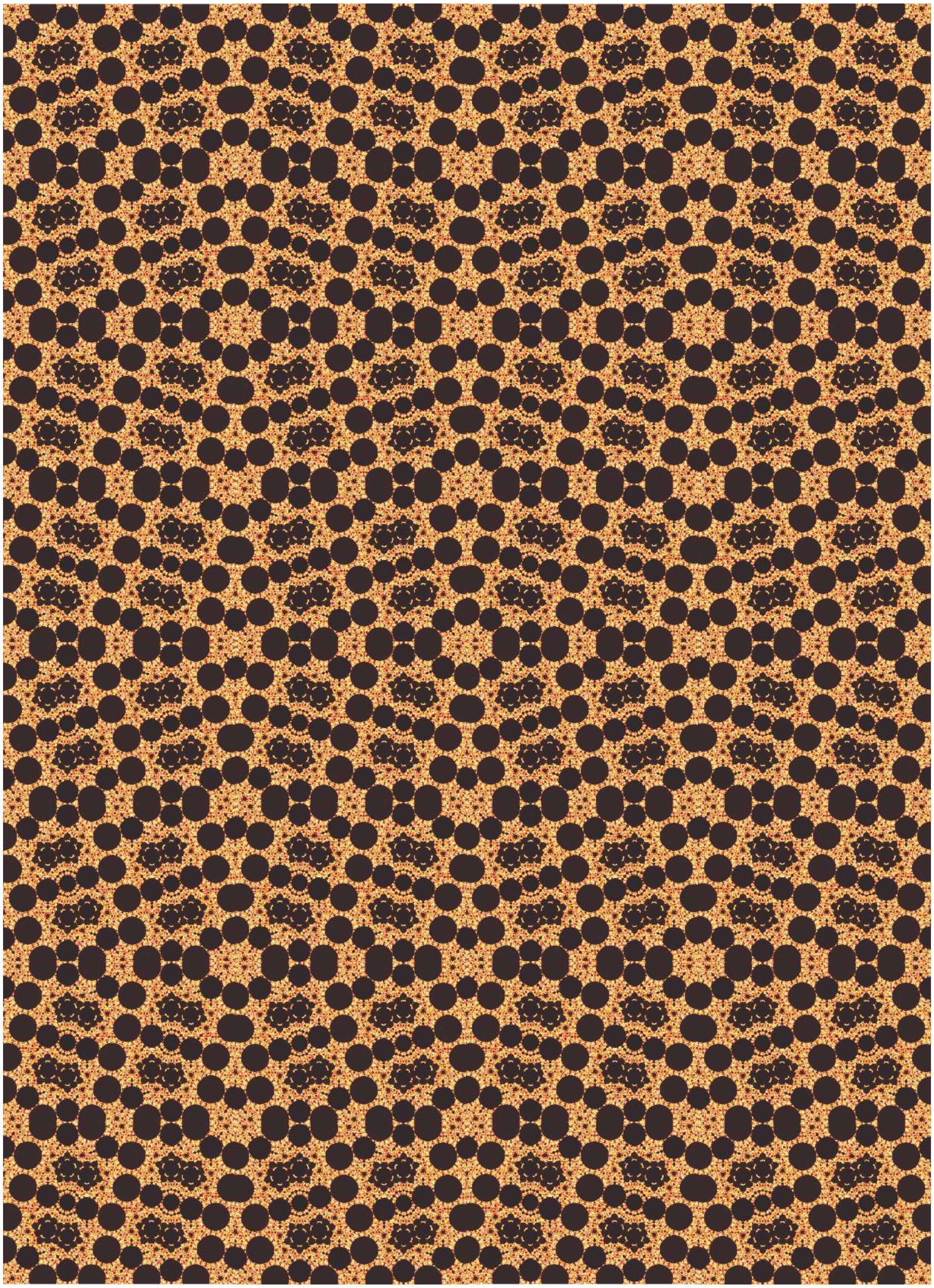


Figure 3.1

Figure 3.2 explores the idea of making a tiling or wallpaper pattern using Jason Smith's N10, R1.84... screenshot shown in Figure 2. It uses a mirroring technique.



With more colors and any of these fractals >> numerous and very intricate tiles. It creates an optical illusion of right, left motion when you scroll it up and down.

Figure 3.2

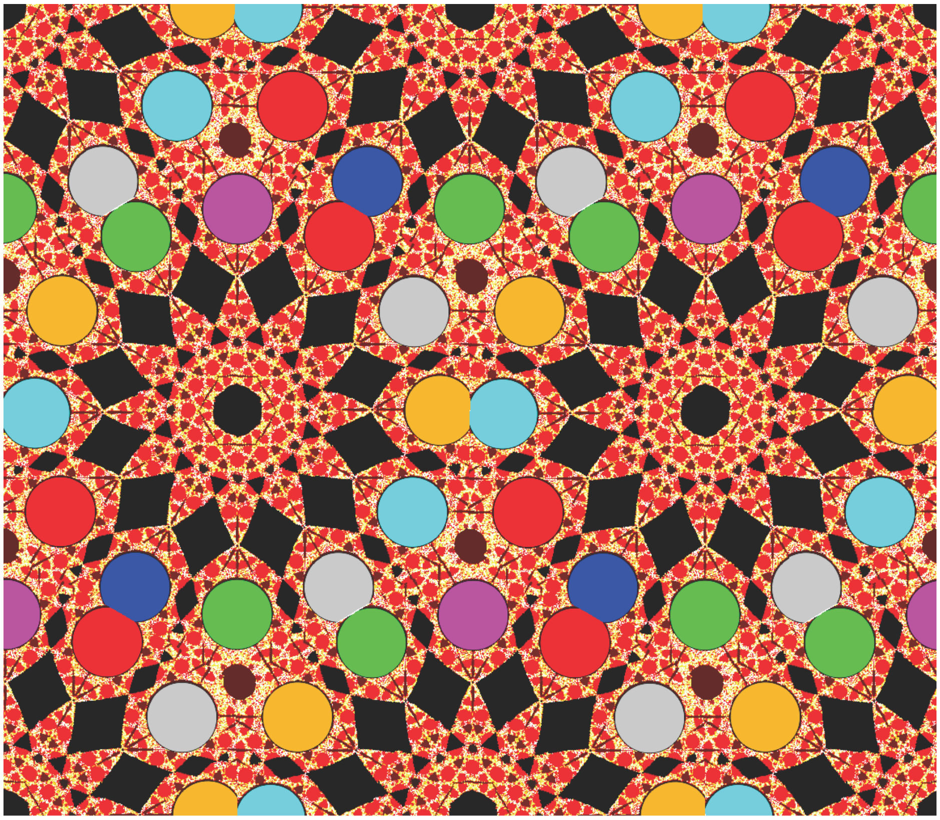
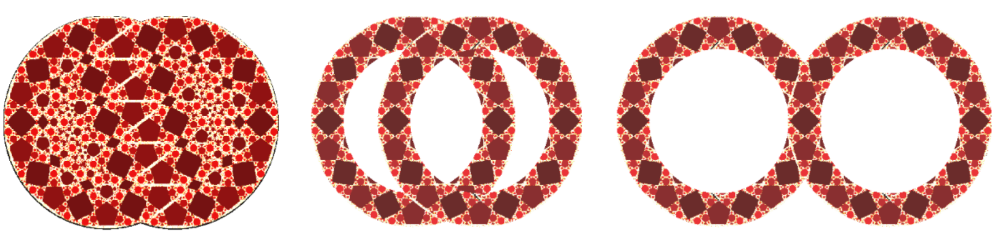


Figure 3.3 is Jason Smith’s Gizmo render derived tile for N12, R2.2799… with colors added. These tiles are really photo shopped fractals but due to the repeating nature of the geometry we can make beautiful crystallographic looking tiles.

Jason Smihts Gizmo render N12, R2.27999

Figure 3.3

Figure 4, left, shows Jason Smith’s N=10, R = 2.98. Center figure shows that an attempt to make it symmetrical is less successful as Rc increases. However, the figure on the right reduces R and starts to show how an annulus geometry might be used to make a symmetrical circle puzzle with squares and pentagons. The puzzle shown is unworkable as a physical puzzle.

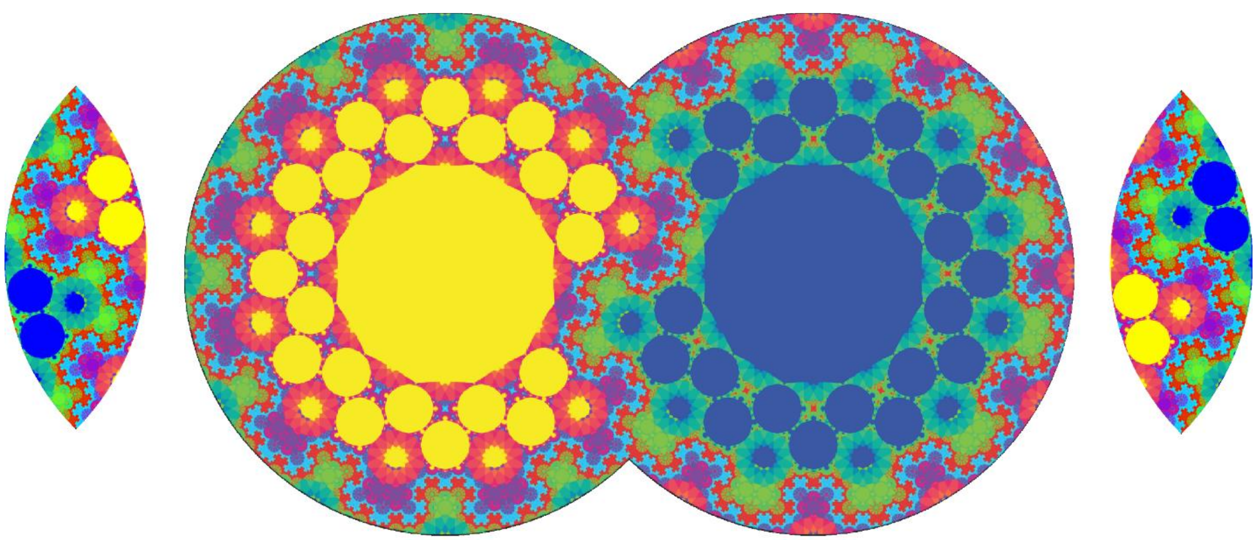


Jason Smith’s Jumbler code for N=10, Rc = 2.98.

Figure 4

Thus the only symmetry these efforts produce is 180 degree rotation, not full circle puzzle symmetry. A different rotation algorithm and perhaps code designed to make the fractals fully symmetrical might produce circle puzzle symmetry but what would it look like? Would the whole thing reduce to dust if Rc is big enough or would there still be nice polygons surrounded by dust? I think such a code would produce very beautiful fractals, for certain Rcx ranges, that could be turned into graphic programs with full circle puzzle rotation symmetry, ignoring any infinite dust of course. These fractals will be of interest to crystallographers. It might show that Penrose tiling's are a unique solution produced only by the N=5 fractals.

Jason Smith's web presentation (using deluxerender.com website) shows a fractal produced by Brandon Enright. Brandon gave me permission to use it here. The Figure 4.1 below shows this N12, R1.446 fractal in several colors. This screenshot reproduction does not do it justice, of course. The two intersection pieces shown illustrate how these fractals have 180 degree rotation symmetry.



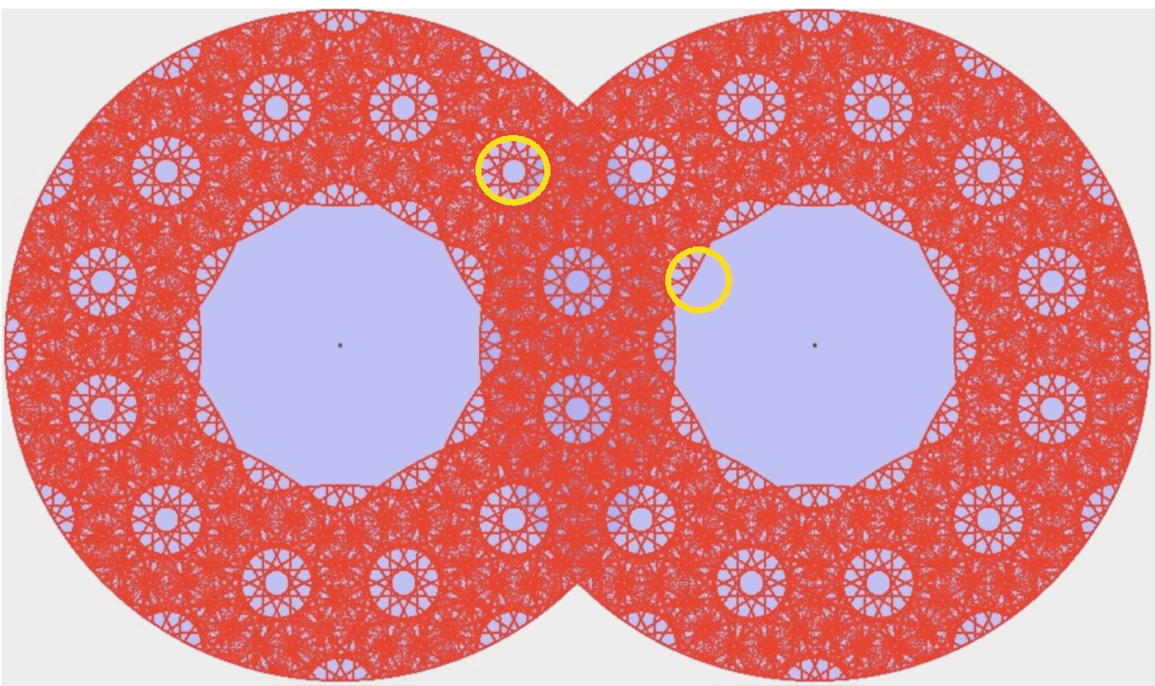
N=12, R=1.446...
by Brandon Enright

Figure 4.1

Bob Hearn's code

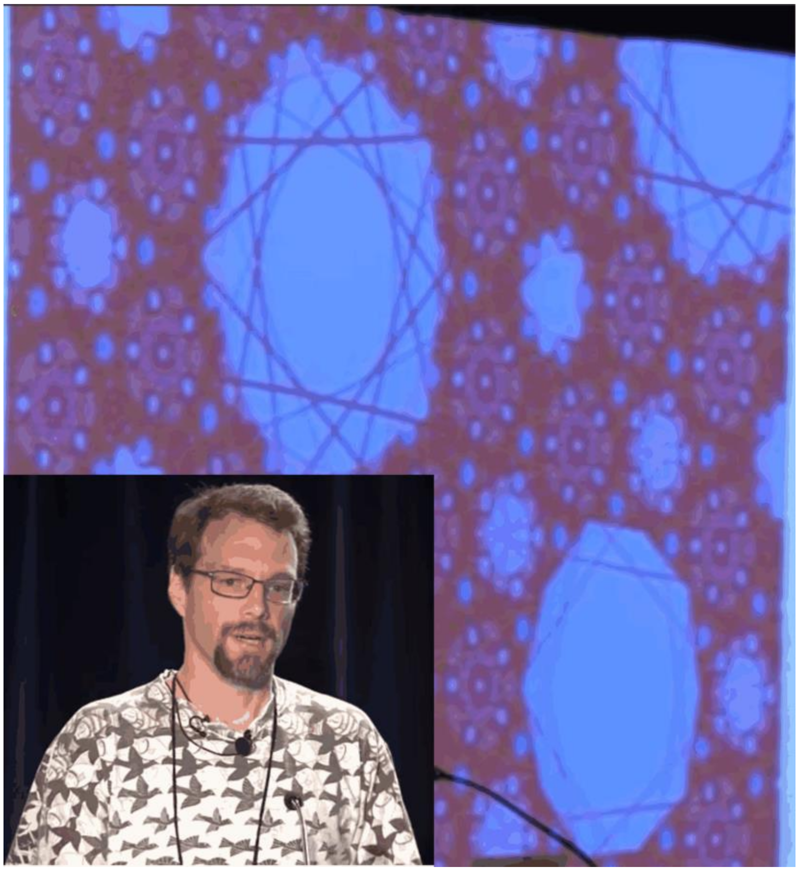
Bob Hearn initially produced code that shows the actual cut pieces of the circle puzzle instead of a point where each new cut occurs. This produced fully symmetrical graphics that are very nice to contemplate. Because it was slow he developed a fast code that just plots a point of each new cut. I am thinking of writing my own code to do full cut symmetry. Here is Figure 23 showing Bob's code for N12, R1.39, which also looks similar to R=(sq rt of 2) as mentioned by Carl Hoff in the twisty puzzle forum link shown in this image. I have included it here to show how it looks only marginally fractal.

http://twistypuzzles.com/forum/viewtopic.php?f=1&t=25752&hilit=gizmo&start=50.



Bob Hearn's full product code N12, R1.39(or 2^1/2) used with permission.
http://twistypuzzles.com/forum/viewtopic.php?f=1&t=25752&hilit=gizmo&start=50

Figure 4.2



Here is a photo of Bob's presentation of his Gizmo code at the 2014 Gathering for Gardner in Atlanta. I was unable to make it to that event. The link here is a video of Bob's 'From Twisty Puzzle Fractals to Penrose Tiles' presentation at the gathering. You can see on the projection behind him the cut pieces of a possible jumbled circle puzzle. His code is able to produce these in intricate detail.

Figure 4.3 Bob Hearn's talk at G4G 2014 in Atlanta, 2014

https://www.youtube.com/watch?v=uHyO7eVun08

Bob's talk was very well done making clear all the variables and how they affect the outcome of the fractals produced. No unbandaging can occur once a critical radius is crossed. You can vary the N and R in each circle and so forth.

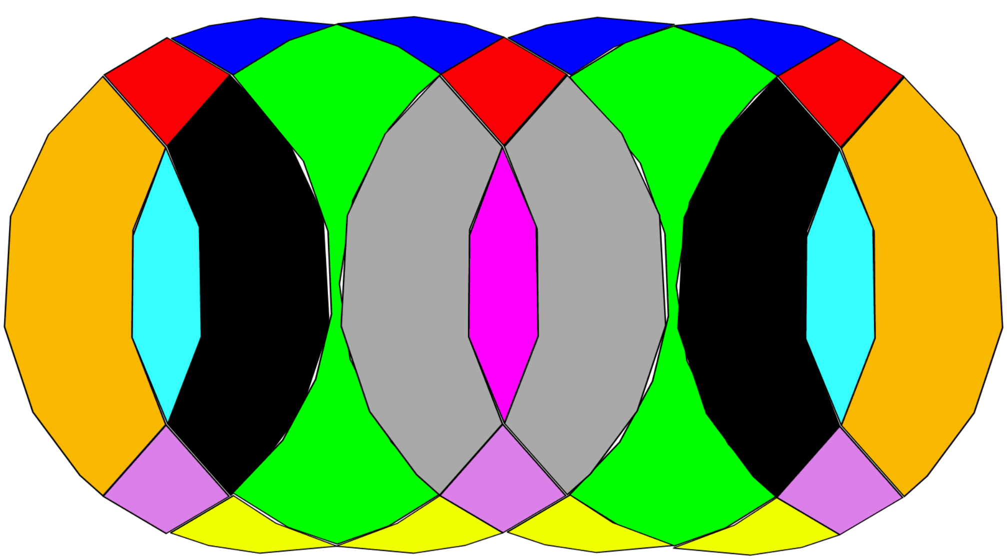
About the fractals

The Twisty Puzzle Forum 'Are Gizmo Gears Jumbling?' produced over 250 (500?) posts. It resulted in a lively conversation (not arguing, just finding out what was going on with these fractals). I will not be able to mention all the contributors. Brandon Enright also wrote his own fractal code and contributed much to the discussion, and a mathematical analysis of Rc calculations. Carl Hoff contributed cutting edge comments clarifying some of the mathematical and geometric considerations. Bram Cohen, the instigator of this, was a key contributor and

Oskar kept this a subject that resonates with his creative energy. Every time I look thru it, I find something new I did not know about. It also resulted in several talks to puzzle groups by Bob Hearn. I think there was also one at Berkeley. The Gizmo posts started mid-2013. Posts have stopped since about December 2016 but the forum is still there, and a great way to learn about how the fractals developed. I am deeply honored that fellow puzzlers give me credit for designing a puzzle that led to fractal circle puzzles, although Bob says it is just on the border of the critical radius.

I still have my own questions. Has anyone proven that once the critical radius is reached are all the following Rcx puzzles fractal? For instance, the forum has a statement that there seems to be unbandaging's at some Rcx's. Can there be finite pieces that are unbandaged and solvable while pieces surrounding them are fractal? Is there any connection to non-periodic tiling's for N >= 7? When does R cause the entire area become dust with no recognizable finite size pieces? Bob Hearn says the pieces are finite in size but infinite in number. At R close to infinity do we always have close to infinite dust, even for N=2,3,4,6, puzzles? Should R ever reach infinity do we get back to a single circle with one piece or do we have an infinite or perfect black circle (as mentioned in CPM)? This is a kind of a 'you make up your own mind' question.

A formula for the number of pieces in a single period binary circle puzzle.



You can have interesting binary circle puzzles. Here is one with two equal periods of the circle centers. This shows that even a binary puzzle can have

Figure 5

different crystal grids. How would this change solution difficulty and orbits over a simple single period one?

Here is a simple 5 circle binary, Figure 5.1, with mixed radius circles. If it were deeper cut it would be more challenging to solve. Deeper cut means to increase the radius of the circle(s) by some amount while keeping the circle centers fixed.

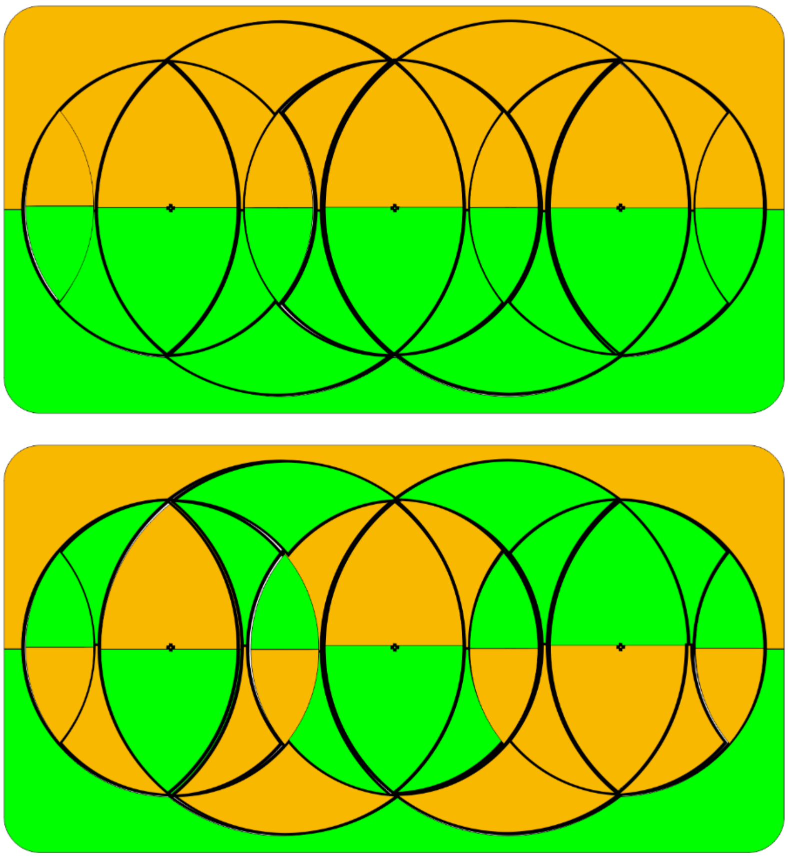


Figure 5.1

The N=2, two circle, single period, binary puzzle is the only example where I have counted the number of pieces in the puzzle for a given R. Figure 5.2, below, shows this formula. One wonders if such formulas also exist for N=3,4,6 and any R. If they do, things will get more involved because as R increases the number of pieces can suddenly be reduced when 3 or more circles cross simultaneously at certain node points. The number of pieces can be increased when pairs of circles become tangent. Two circles share the same node point when tangent. However, I did not allow for tangency in the N=2 formula, but it would be easy to do so by having a formula for whenever R produces tangency and the modified piece formula when this happens.

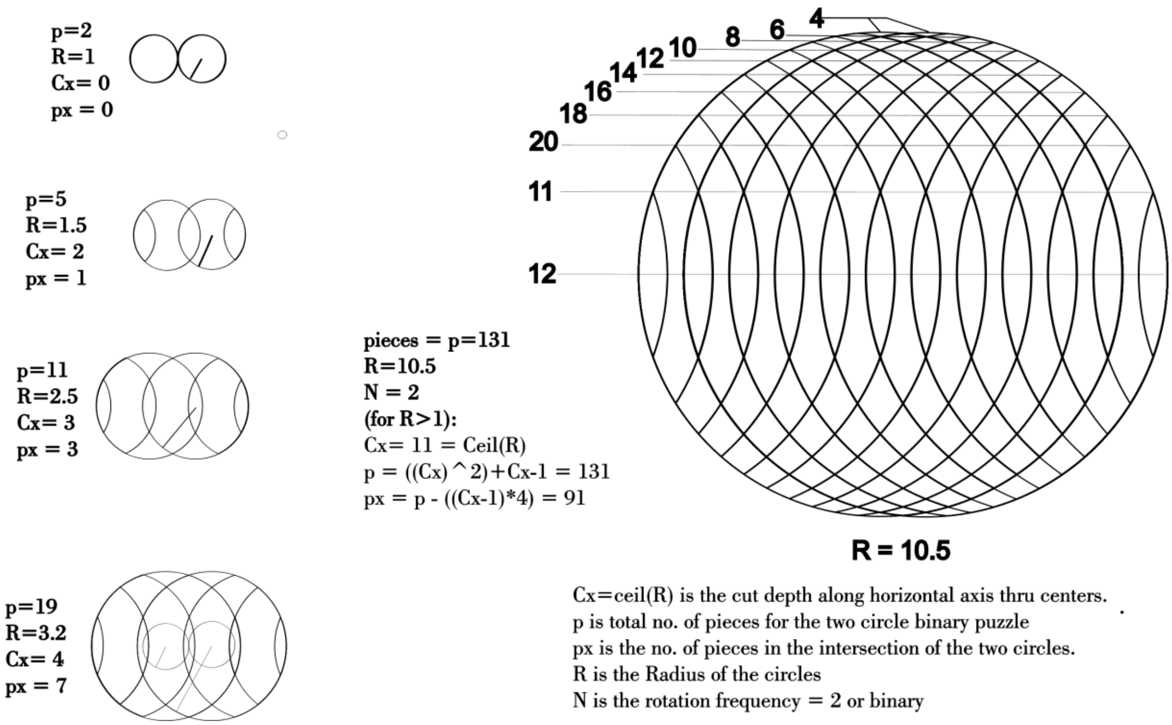
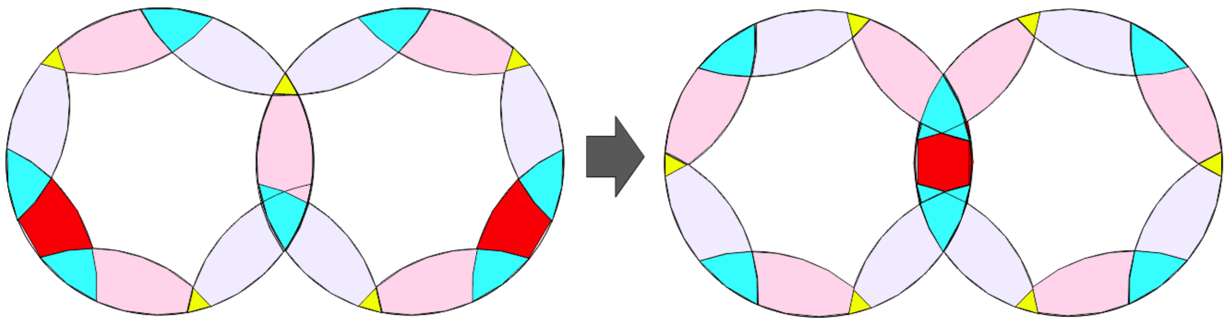


Figure 5.2

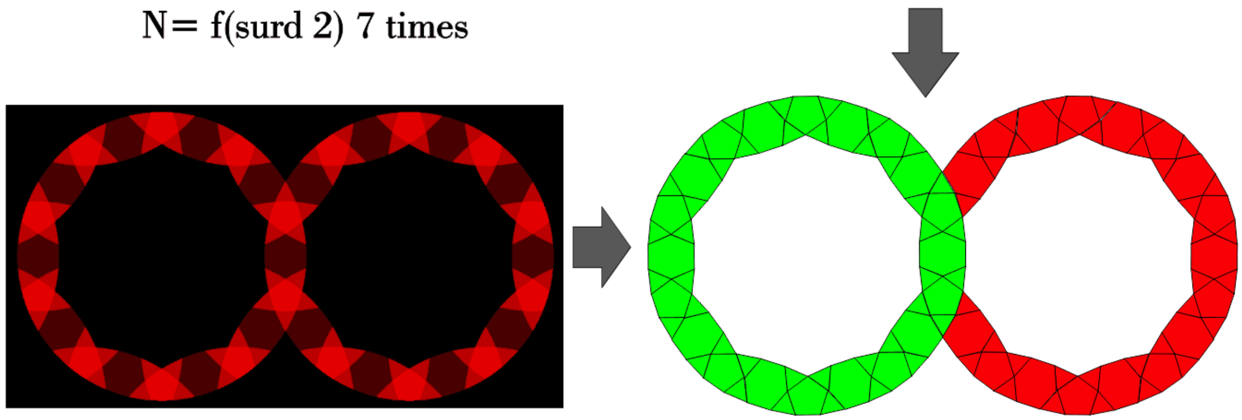
A circle puzzle derived both from a surd and a Gizmo render.

Figure 6 shows a circle puzzle gotten by using N=infinity, root 2 rotation and Jason Smiths N=10 render screenshot just before it goes wildly fractal. N is infinity since the root 2 rotation never closes on itself but forever fills the circle with new cuts. The bilateral symmetry allows creation of many such highly bandaged surd puzzles. We will show more of the surd 2 below.

It is interesting that the surd 2, with 7 clock ticks, pattern can be used to approximate the N=10 render by some photo shop trickery. It shows how puzzle design has to do with both art and math at the same time.



N= f(surd 2) 7 times



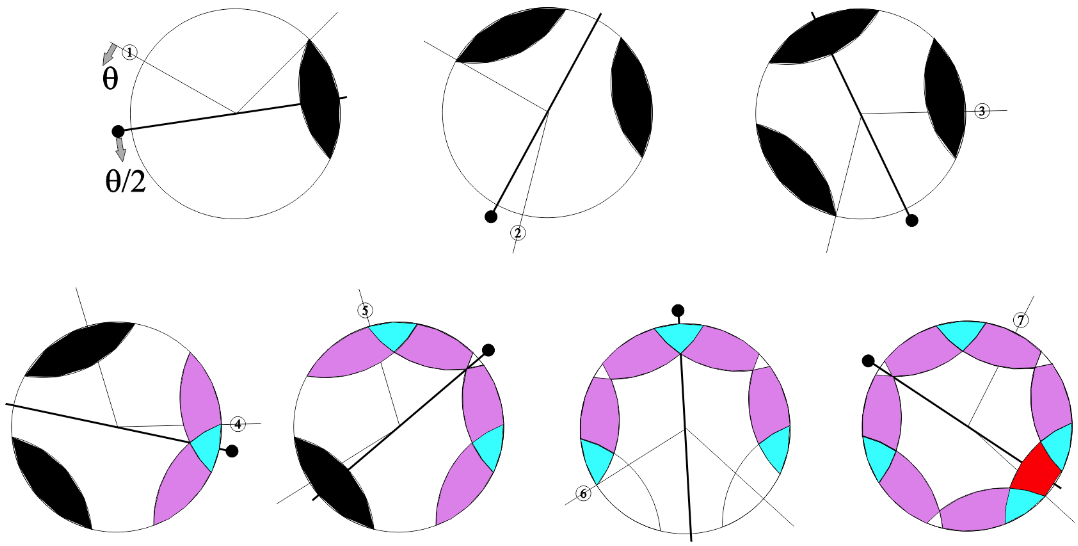
Jason Smith's N=10 Gizmo render.
R=1.17, N=10, θ=36 deg

Three curved polygon shapes,
R=1.17, N=10, θ=36 deg

Figure 6

A surd 2 infinity clock

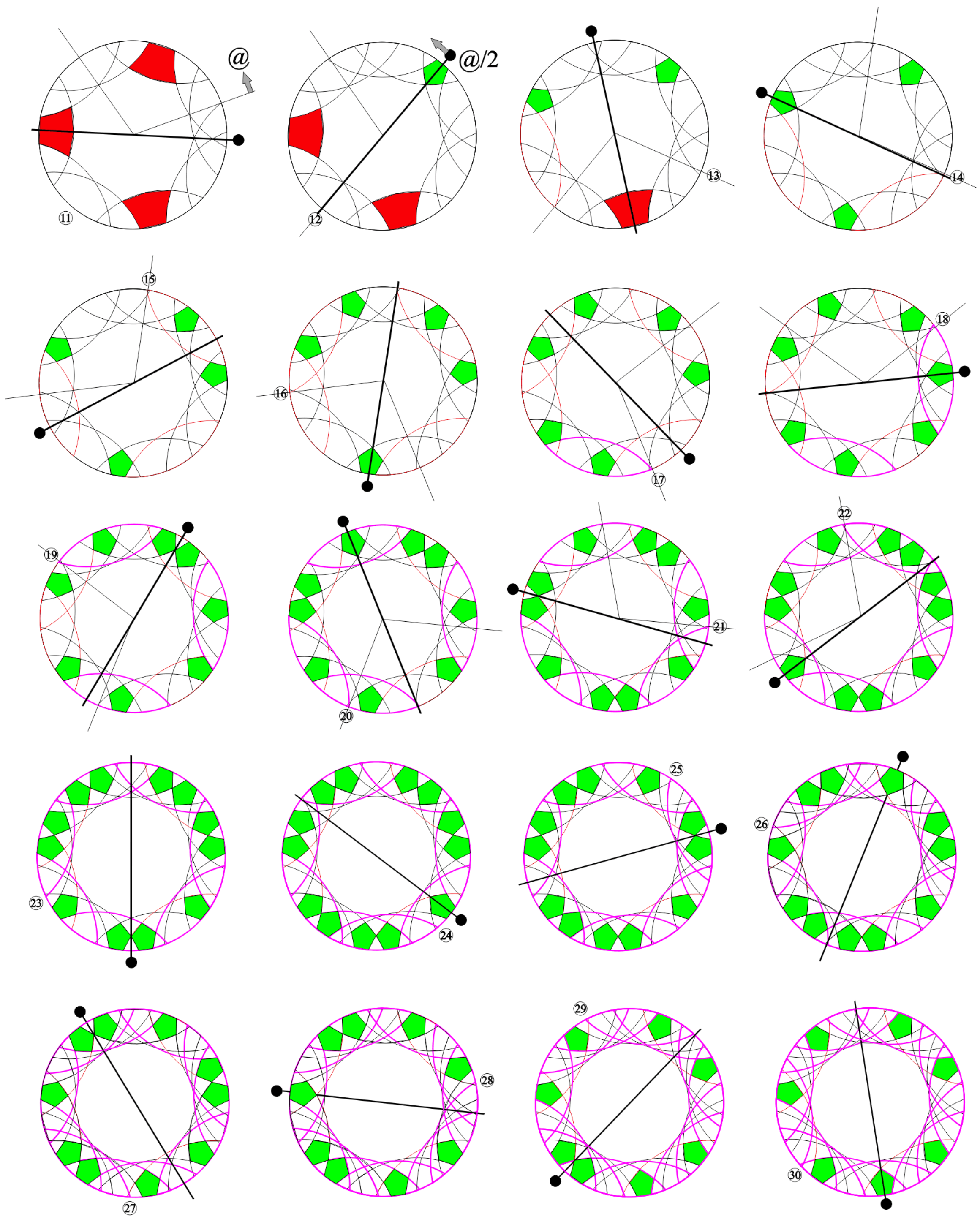
We can explore what happens with the surd 2, or N=infinity idea. In Figure 7 we use this to run the clock for 7 ticks noting that every pattern has perfect bilateral symmetry. This is common knowledge since every rotation is an equal number of degrees. The interesting thing is that it produces many perfectly symmetrical and equal pieces as we continue the clock ticks. One wonders if more complex unbandaged circle puzzles could be designed using this process.



A Square Root of 2 Infinite Symmetry Clock, 1 through 7 ticks of the clock.
N = infinity, rotation = θ = 360 - 360/(2^1/2) = 105.445... Symmetry rotation = θ/2

Figure 7

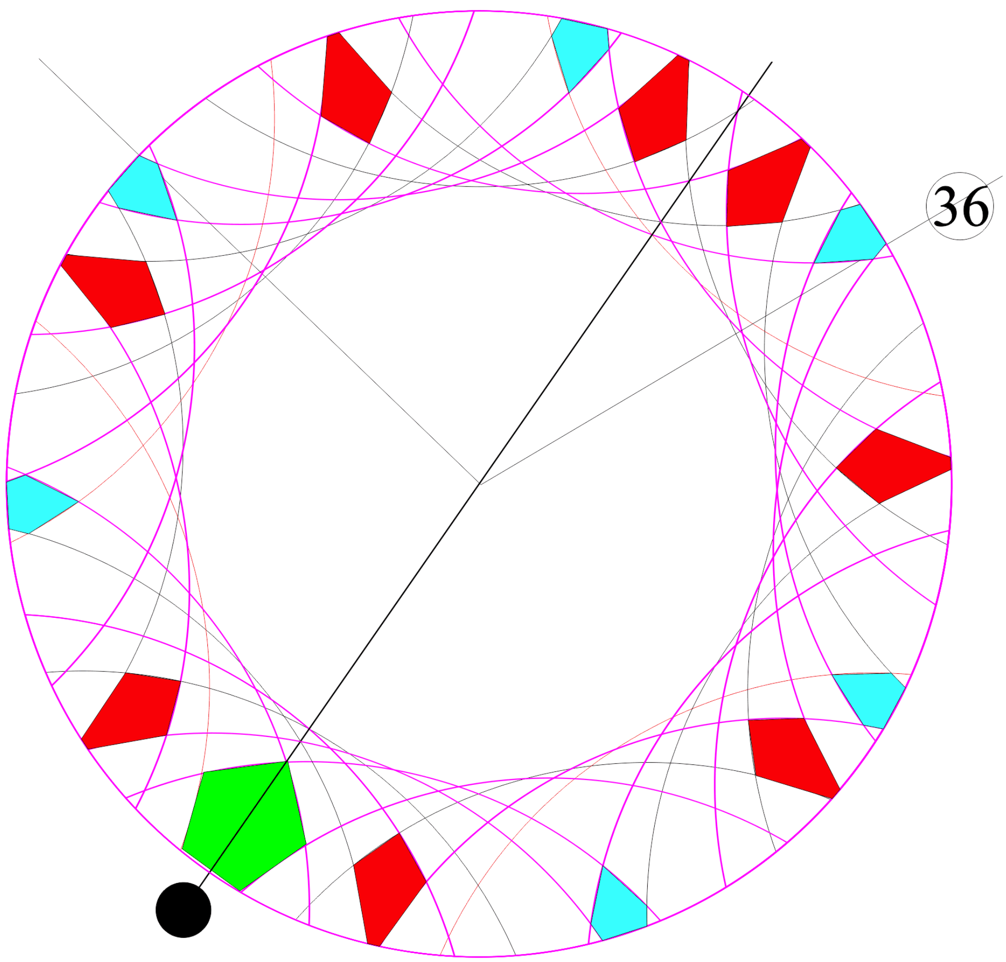
Figure 8 below takes the ticks from 11 thru 30 showing how pieces start as one new piece then increase one at a time to a maximum then decrease one at a time and finally disappear only to start all over with a similar smaller piece as the tick's progress. **As the ticks progress the pieces get smaller so that there will be more pieces of the various kinds since they make smaller targets for cutting destruction. It ends up being a remarkable sequential counting up and down of identical pieces in very a specific, mathematical, and orderly manner.** This behavior is very similar to the Gizmo fractals behavior. As R increases patches of geometry develop, increase to a maximum then decrease and finally disappear, only to start all over again. The big difference is that each new piece in our clock counts up one by one to a maximum in each cycle then counts down one by one to zero. It is as if N based on a surd goes thru all the cycles of Gizmo fractals in one fractal, at least in this one aspect. If cuts were reduced to points as was done to produce the Gizmo fractals can surd clocks produce fractal images?



The Square Root of 2 Infinite Symmetry Clock
N = infinity, rotation, $\theta = 360 - 360/2^{1/2} = 105.445...$
Symmetry rotation = $\theta/2$

Figure 8

Finally Figure 9 shows tick 36 enlarged to better illuminate the individual pieces produced. Note that if a piece is non-symmetrical its mirror image appears across the bilateral line of symmetry. This is the last tick where the green pentagonal piece appears. At tick 37 it disappears, cut thru by tick 37.

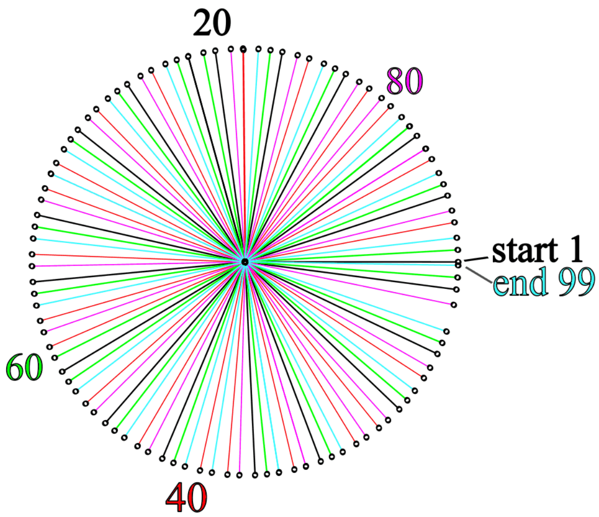


Bilateral Symmetry Clock, t=36. Space must expand as
the cuts increase to keep piece size standard.
New pentagonal symmetric particles are created as cuts increase.

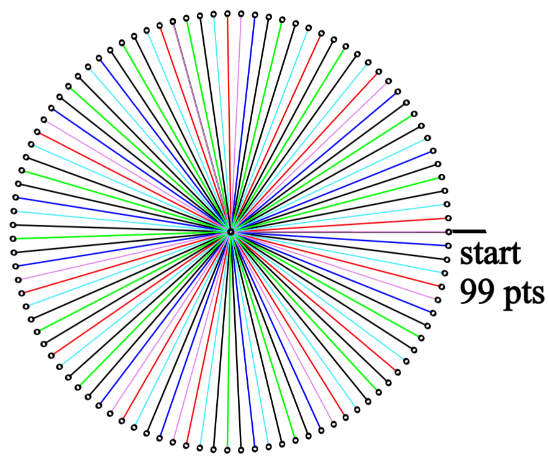
Figure 9 A single circle 'Clock Fractal'

Because of bilateral symmetry a surd clock, at some chosen tick and piece coloring, can be a graphic puzzle. Clicking on any arc would transpose the pieces inside the arc with its symmetrical mirror image. Then after a few clicks the mixed-up pieces could be solved by more clicks.

Using a radial line and N based on f(surd2) rounded off to 3 decimals Figure 9.1 is produced. The left most figure has been gotten using the nearest rounded place while the rightmost figure adds 0.01 to theta and this causes the radial lines to repeat after 99 ticks while the non-biased rounding is able to remain random looking. Bilateral symmetry, in both cases, is maintained at each tick of the clock.



irrational N=360-(360/2^1/2)
=105.445.... min. deg.
The spacing of pts varies.



Add .01=105.455 instead of 105.445. Points repeat at approximately 99 rotations, 99x105.455/360=29.0001

Figure 9.1

Polygonal puzzles and tessellations derived from circle puzzle pieces

Take the pieces of a circle puzzle and make all the curved edges straight to produce a set of polygons. Figure 10 shows this process for a N=6 and N=4. Surd clocks are an open invention. You can experiment with them. For instance, how does increasing R affect the number and shape of the pieces? Is there a formula for the number of pieces? Is there a formula for the number of different kinds of pieces and the number of identical pieces? Knowing how pieces count up and down as time passes such a formula might be possible. You could also experiment with arcs having straight edges and with arcs defined by some equation other than a circle. Many other graphic and mathematical investigations are possible.

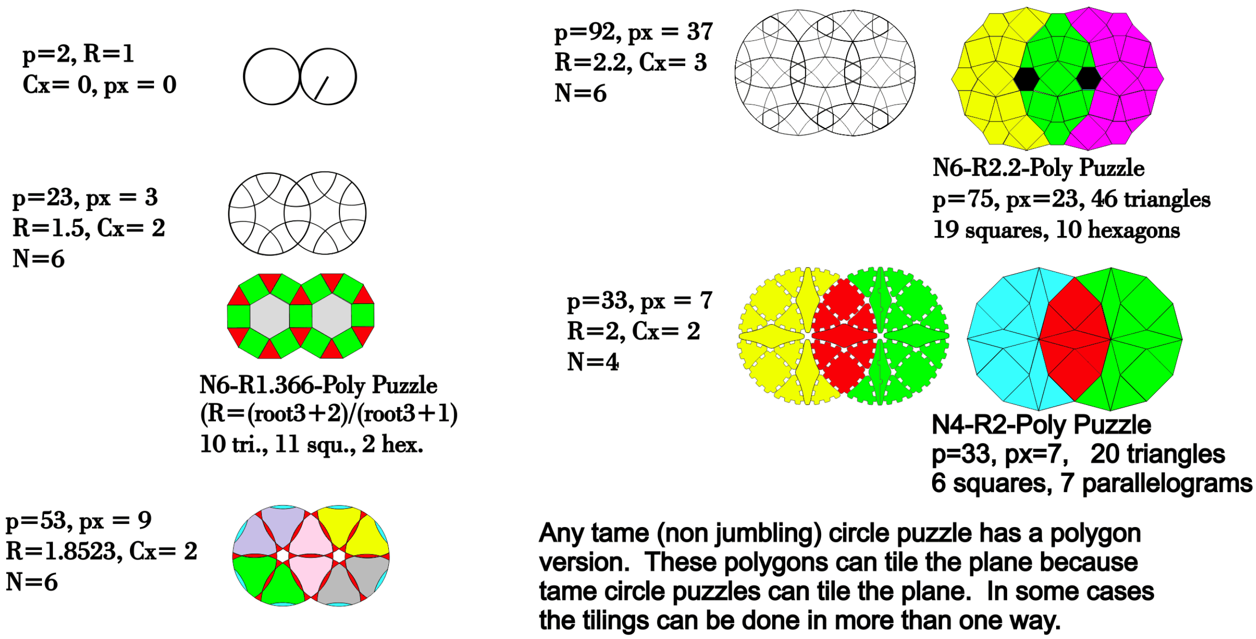
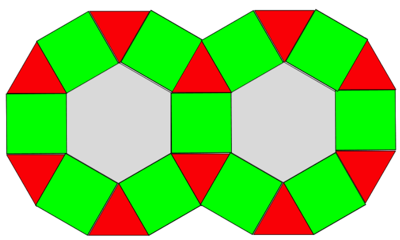


Figure 10

In Figure 10 p is the number of pieces, px is the number of pieces in the intersection, R is the radius, Cx is the number of circles and N is the rotation frequency so that for N=6 the smallest rotation is 60 degrees = 360/6. When making the straight edge polygons we try to make them regular polygons when possible. In some cases, the polygons derived can tile the plane in more than one way. In other cases, only the circle pattern is possible. For the R= 1.8523 case there was no attempt to reduce to polygons since some of them are concave. Of course we could easily reduce to convex polygons by dissecting the concave ones.

Figure 11 below shows a tiling for the N=6, R=1.5 case which produces a triangle square and hexagon. These can tile the plane in an infinity of different ways.

The rightmost figure below is a beginning of a random tiling, nothing that is new here, but it illustrates the connection we want to make.



N6-R1.5 as regular polygons

Problem:
Using 2 or more different convex regular polygon tiles that can tile the plane in a periodic way: is a random tiling always possible?

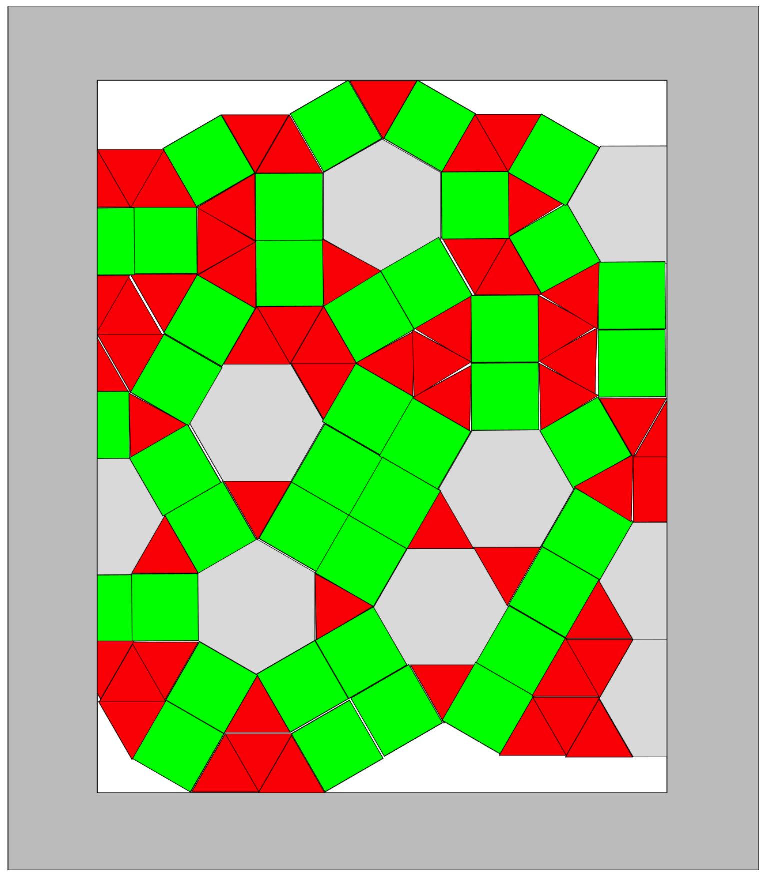
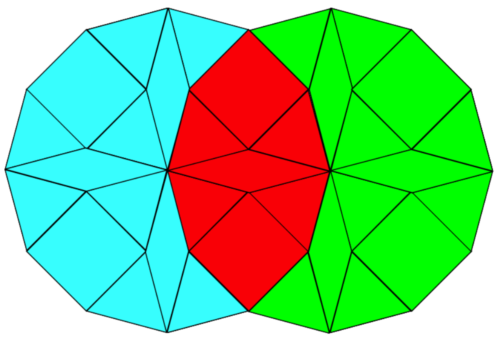


Figure 11

Figure 12 shows the start of a random symmetrical tiling using polygons from N4, R=2. It is shown without proof that it shall be possible to continue tiling from this start in a random manner making corrections when the tiling process presents a stopping point.



N4, R2 as polygons

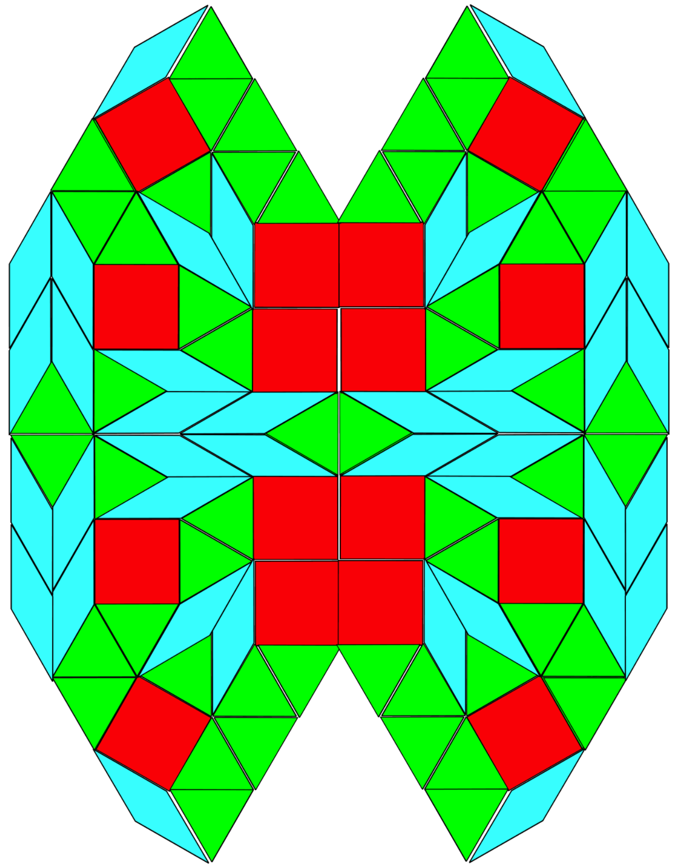
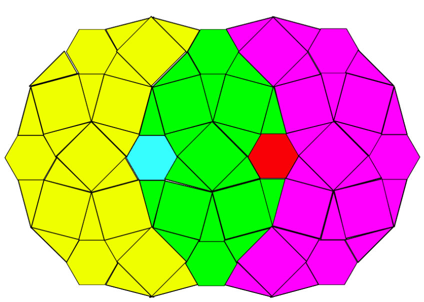


Figure 12

In Figure 13 below we show N6, R2.2 puzzle reduced to isosceles triangles, squares and hexagons. These polygons can also tile the plane randomly. The random aspect is more restricted but still works.



N6, R2.2 as polygons

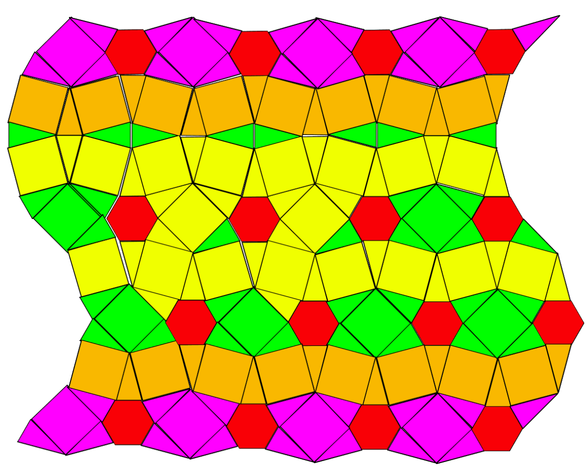


Figure 13

Here is a graphic circle puzzle made with pentagons, N10, R1.475, Figure 14. While not a physical puzzle it could be programmed as a graphical puzzle. The decagon pieces with arrows indicate each rotation.

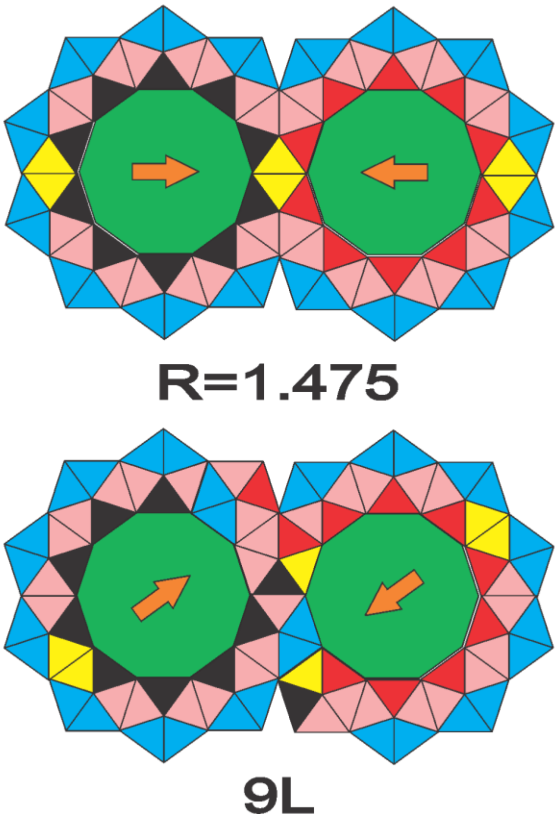


Figure 14

Using the polygons of Figure 14 and adding a triangle and parallelogram we can make the tiling in Figure 15. Is it possible to make a random or semi-random tiling with it?

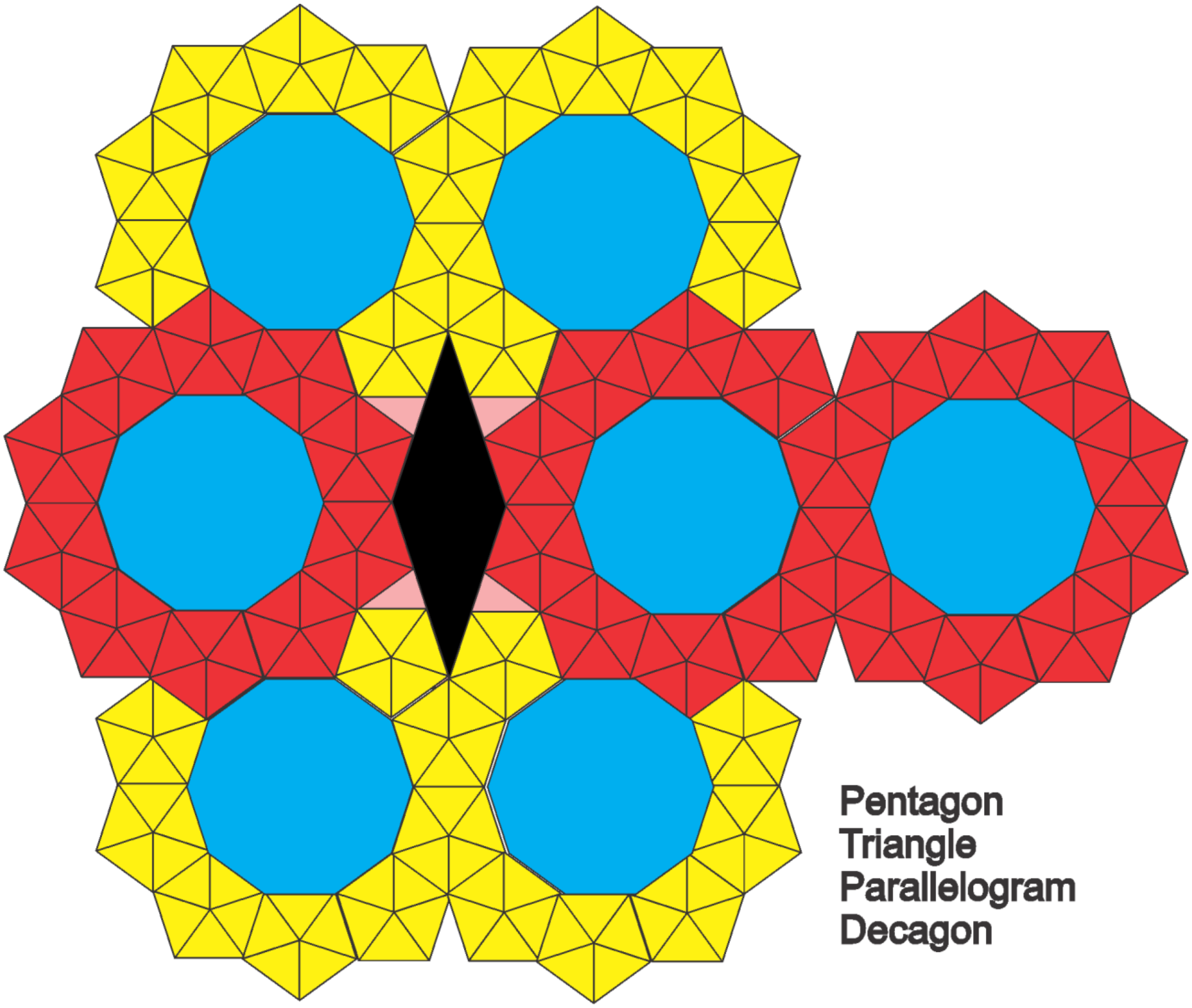


Figure 15

With N12 and R1.32 we can make Figure 16 as a graphical puzzle design. No solution algorithm has been devised for these graphical twisty's but it should be straight forward to do so. They are sort of like gears where the teeth of the gears constantly rearrange themselves.

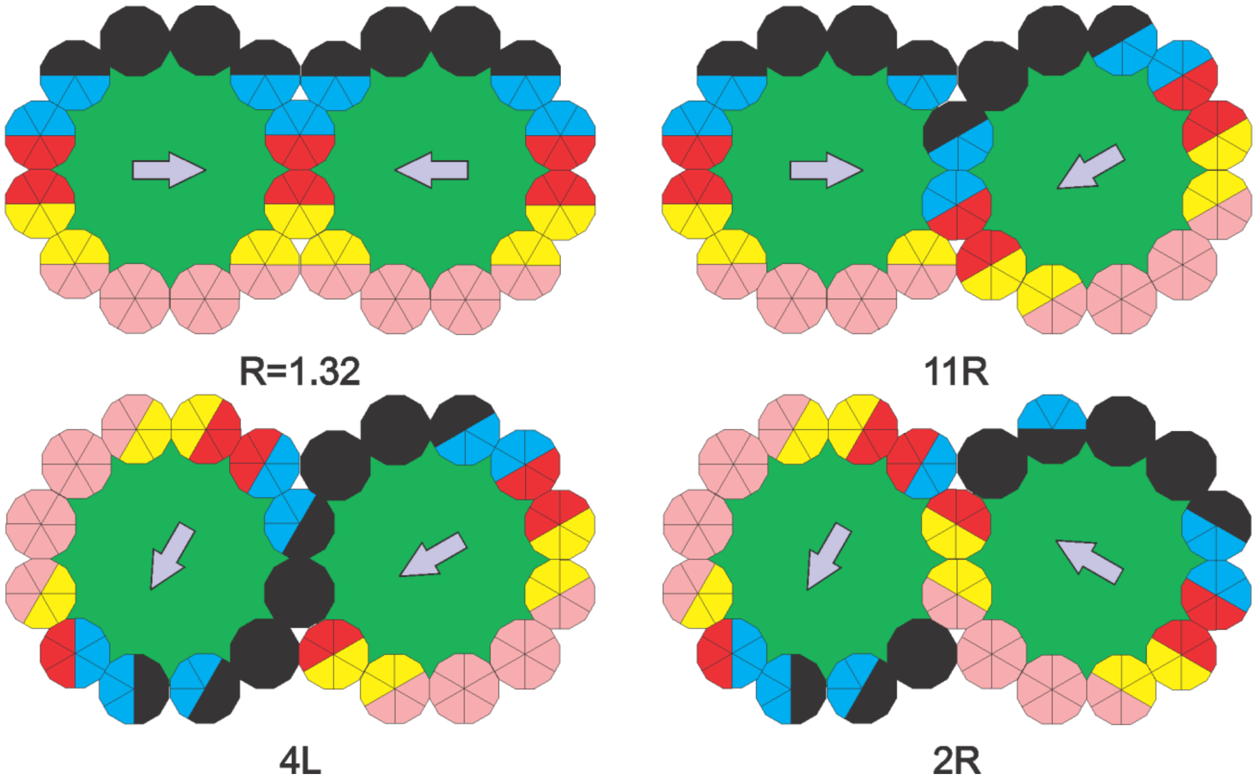
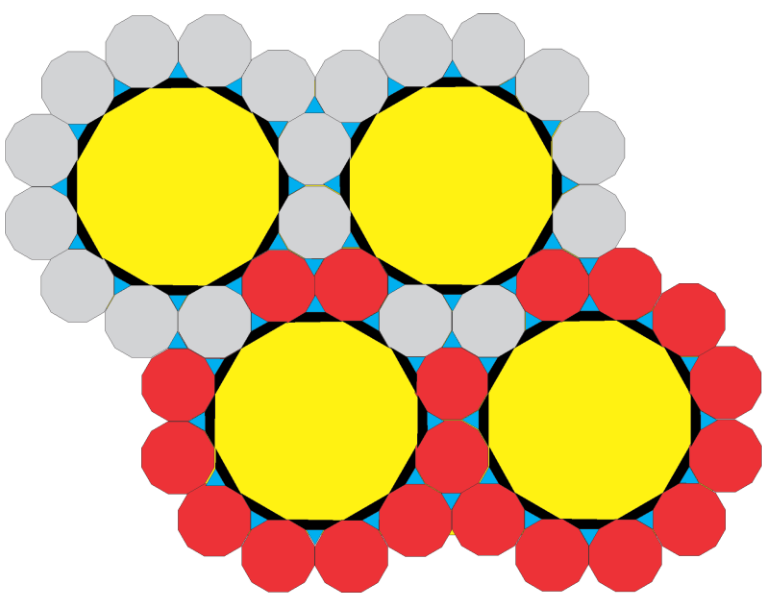


Figure 16

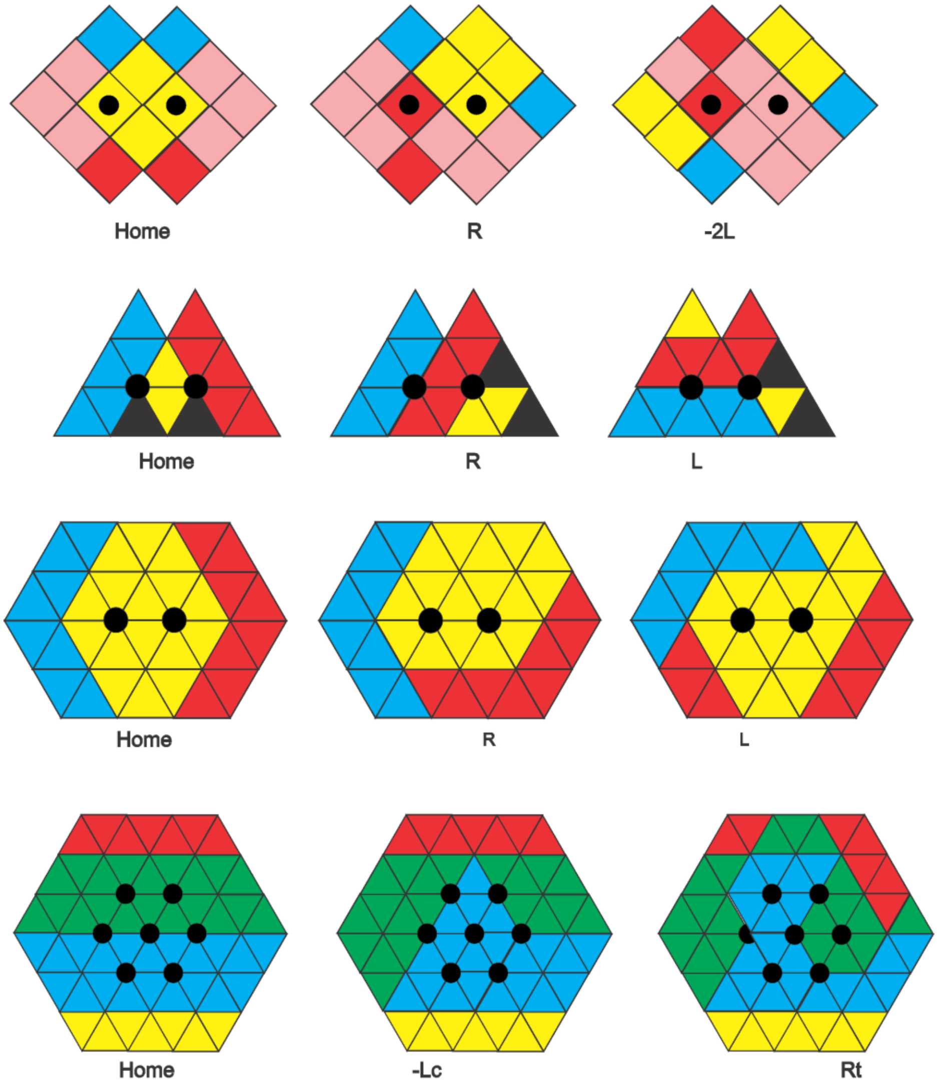
The puzzle in Figure 16 can be used to tile the plane with polygons as shown in Figure 17. The question again is can you do it in a random manner as well?



Dodecagons can fill the plane with 3 polygons
and an isosceles trapezoid.

Figure 17

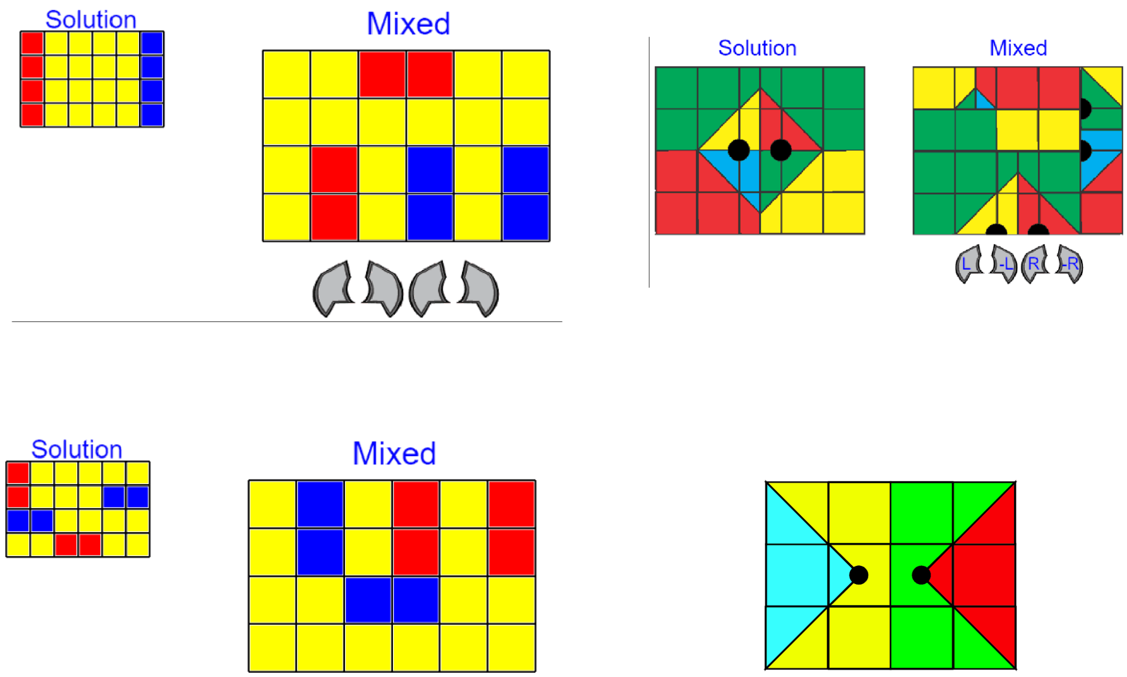
Mono-polygon puzzles



Polygon ‘circle’ graphics puzzles of various designs showing result of 2 rotations

Figure 18

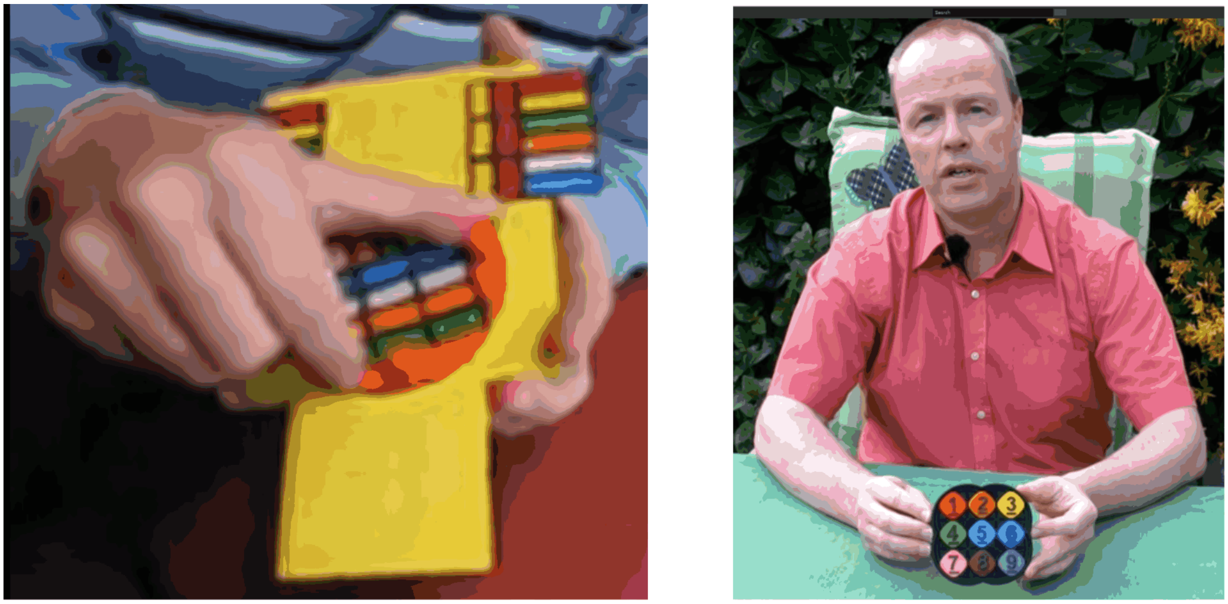
Figure 18 shows several different examples of polygon graphic type puzzles called mono-polygon puzzles. Mono meaning that each puzzle has only one kind of polygon piece. I thought the idea might be new but Oskar van Deventer gave me a link to a pair of physical puzzles he developed based on suggestions from friends. Link https://www.youtube.com/watch?v=YyKrvES8Sb4 is to Bram’s Square which uses both sliding and rotary motion. Oskar believes it could implement any of my designs, but I am not sure about the triangle and hexagon ones. Oskar developed another one suggested by another friend and you can see



Four Twisty squares games https://www.puzzleatomic.com/GAMES_2.htm

Figure 19

it here: https://www.youtube.com/watch?v=pNzE-B57G9k This one is called Turn Four. Below are some screen shots of these interesting puzzles.



Bram's Square and Turn Four Implemented by Oskar van Deventer

Figure 20 (screenshots used with permission)

All these mono twisty puzzles illustrated here are pretty much self-explanatory but that does not mean they are too easy to solve, some can be challenging. I can see where the slide and rotate one could be quite a challenge (from my experience with my Palette puzzles) but one never knows until one tries it. One interesting thing to do is analyze the orbits of the pieces. For instance, if the intersection is deep and the square array has mutually prime dimensions each piece can visit every cell of the puzzle, but this may not make it more difficult to solve.

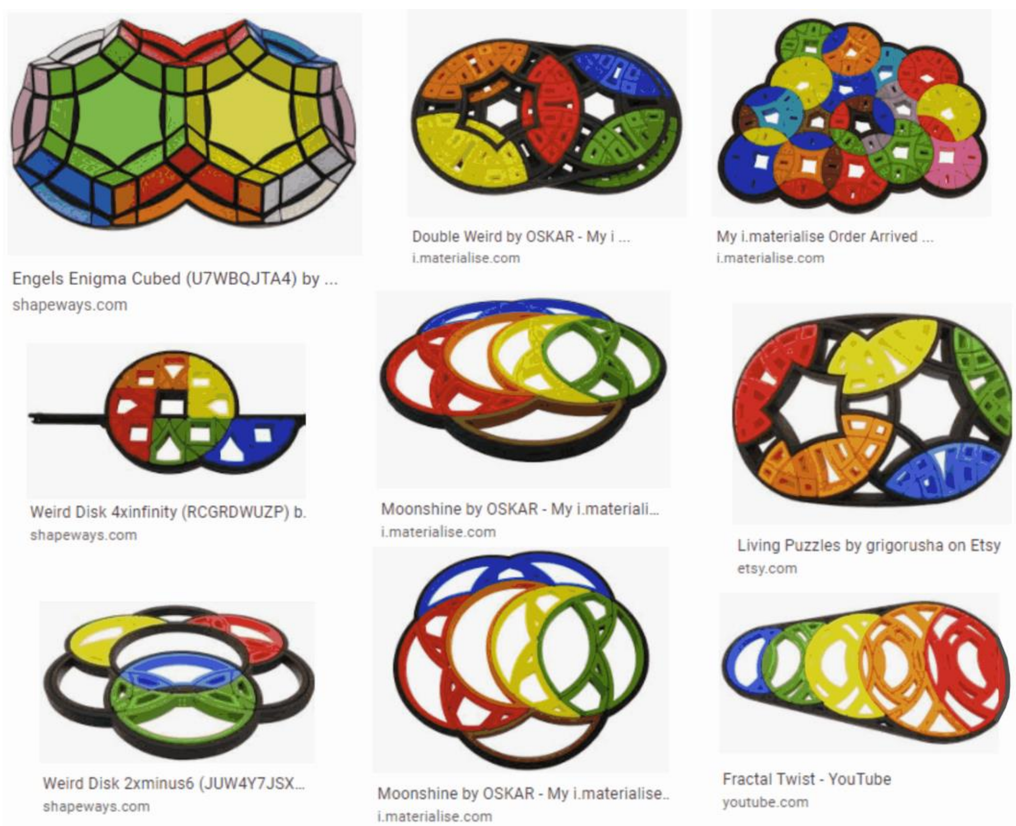
Review of a selection of the latest mostly planar twisty puzzles

Figure 21 below shows five circle puzzles by Very Puzzle using a bandaged kind of pentagonal design. These puzzles only show a partial symmetry but are interesting examples of the wild design possibilities with both planar and 3D twisty puzzles.

Semi-symmetric(bandaged) circle puzzles by Very Puzzle

Figure 21 http://www.verypuzzle.com/

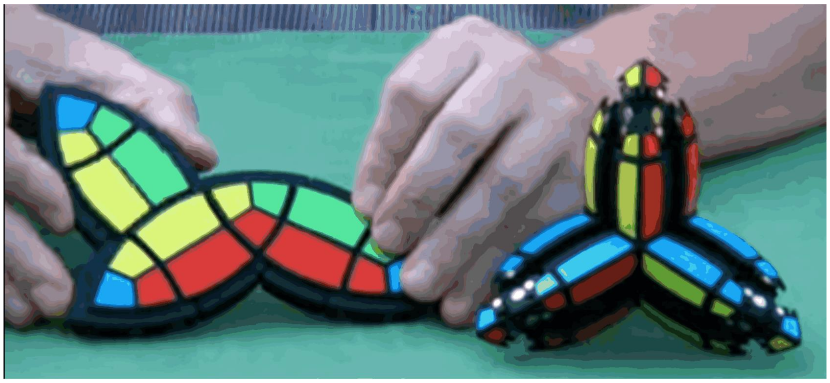
Oskar is a talented and creative designer of all kinds of puzzles. In Figure 22 some of his new designs are shown. My original CPM showed many designs, but I believe all of Oskar's are new ideas. I think his Double Weird and Fractal Twist in the figure could be imagined in some way to evolve from my infinite conical intersection puzzle shown on the cover of the original CPM. For instance, one of the cones could be inverted to get something like the Double Weird. But his Fractal Twist is quite amazing and would need an extreme warp of a cone. I was honored to see his Engel's Enigma Cubed, a 3D morph of Engel's Enigma. You can see some of these designs by going to Oskar's videos on Google.



Recent circle puzzles designed by Oskar Van deventer,
a prolific designer of twisty and other puzzles.

Figure 22 https://i.materialise.com/forum/t/double-weird-by-oskar/1042

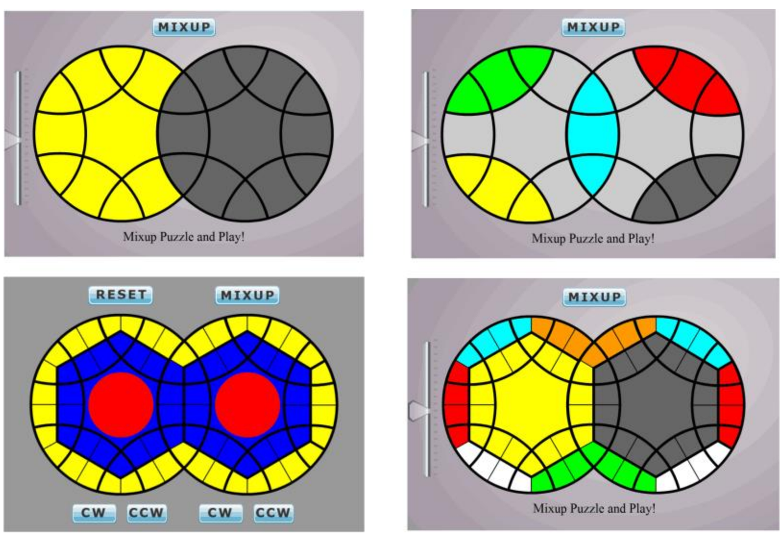
Dust Twist, Figure 22.1 is a unique puzzle by Oskar (and a personal favorite of mine) that uses pieces from a circle puzzle to create a partial circle puzzle using a circular sliding action. It is quite a nice idea, thinking outside the box. It is an example of a new surprising design that makes puzzle designing and collecting a rewarding, fun activity. Along with Dust Twist the video link shows Oskar's 3D version of Dust Twist using the slide action and magnets to align pieces after rotating them.



Dust Twist Puzzle by Oskar van Deventer

Figure 22.1 https://www.youtube.com/watch?v=Fw35w1gZJa8

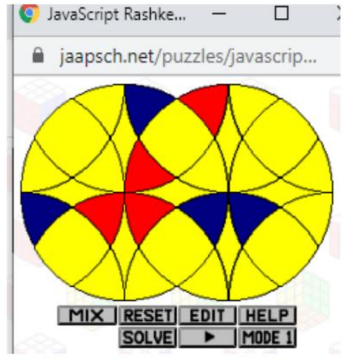
Figure 23 shows four free Magical gears games that can be played on my website Puzzleatomic.com. If you want the computer to solve it will simply reverse all the moves, you have made so it may take some time. The graphics are not fancy, and you can enjoy on cellphone, tablet or pc without lugging a physical puzzle around.



Free games current on Puzzleatomic.com

Figure 23 https://www.puzzleatomic.com/GAMES_2.htm

https://www.jaapsch.net/puzzles/rashkey.htm
Rashkey puzzle invented by Oleg Raschkov
patetented 9-1999 DE 29.904.348 U



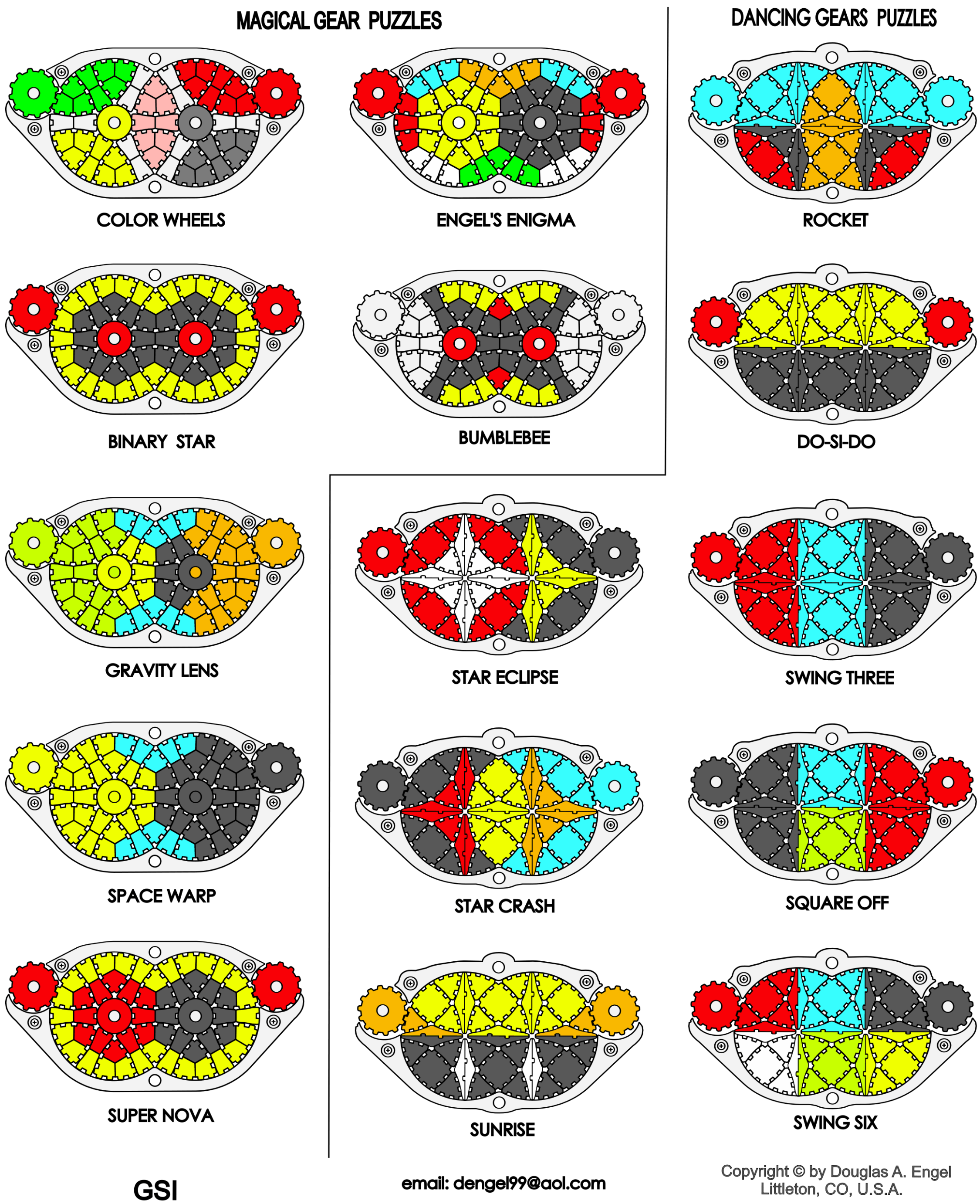
https://twistypuzzles.com/cgi-bin/puzzle.cgi?pkey=1528
Hog Wild Double Think Binary Ring
Inventor Sergei Abramenkov, 2010, from Jaap's puzzle collection. Produced by Hog Wild

A 'Petal' circle puzzle purchsed at an IPP about 12 years ago.

Figure 23.1

Three Flatland designs

The top right and left figures are the Rashkey design by Oleg Rashkov in 1999. At right is Jaap Scherphuis graphic Rashkey game, free on Jaap's twisty puzzle website. The Rashkey is an improvement of my Dancing Gears in using a radius of 1. If l had done this with my gears I believe it would have solved the tendency to torsion lock. The bottom right puzzle was purchased at an IPP about 12 years ago. At this time, I could find no more info on it. I recall that it was from Hungary and called Petal or something like that. There is an email on the back 'Arunmandorla@gmail.com' also a name Maya Jena K. It has a bit of friction to operate but otherwise very sturdy. It is almost identical in both size and design to an epoxy cast one, about 1989, I sold a few of, except mine was housed in container. This one has a surrounding border so pieces can be gripped from both sides. The bottom left figure by Sergei Abramenkov, 2010, is also courtesy of Jaap's collection on the Twisty Forum. It looks like a stack of circles than can be rotated separately. From the picture the concave pieces look rotatable about the tori axes. It is no longer available from amazon. It was called 'Hog Wild Double Think Binary Ring'.



Puzzles in acrylic produced by laser cutter early 2000's

Figure 24 Puzzleatomic.com

In Figure 24 we have seven of the Magical Gears examples and eight Dancing Gears. I made several out of acrylic but often they needed tuning because the laser melts the acrylic when it cuts it so various effects of fit, temperature, humidity, age of plastic, etc. can affect the accuracy of these when made of acrylic. Several years later having so much trouble producing these I decided to try using 1/8 inch, or 3mm wood paneling. It is a low-density Masonite type of paneling. Almost immediately many of my problems went away. The wood is lighter, easier to cut and the pieces can be hand colored. The great advantage of the gears is that your hands never touch the pieces, and you have control of both circles. Another plus over 3D is that there is no need to turn the puzzle over, all pieces are visible. The big disadvantage, and the reason for all the colored versions, is that difficulty is high, having only two circles to unmix with. With fewer colors' difficulty decreases. I no longer make the Dancing gears. The technique with wood laser cutting produces all pieces in one cut, no glue or screws required, just a single pass over the border contact curves with fine sandpaper, and a little paste wax on same, then assembly. You have a puzzle you can carry around in your pocket. Circle puzzles are a great way to teach some of the properties of group theory, as everything remains visible and orbit diagrams are easy to show, etc.

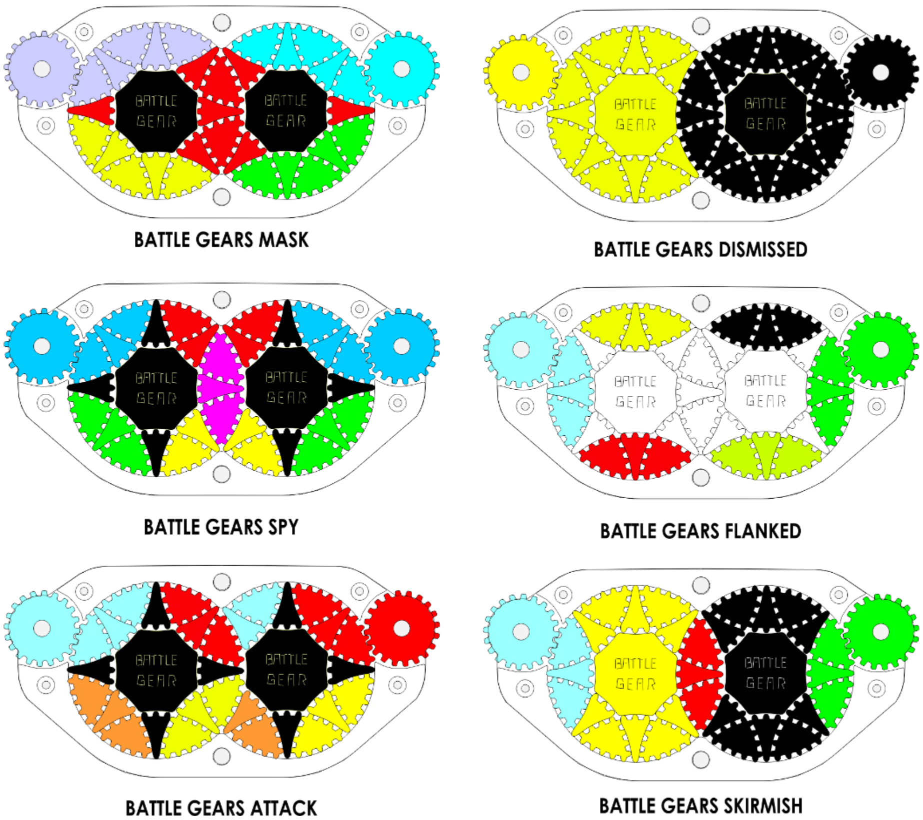
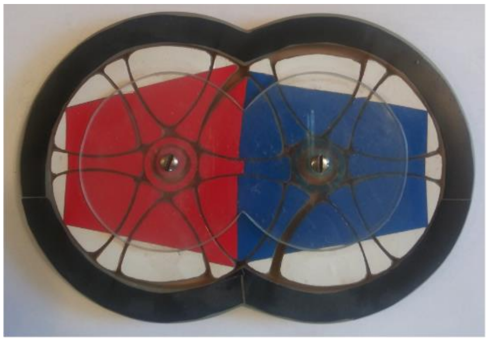


Figure 24.1

Figure 24.1 left shows six versions of Battle Gears a bandaged puzzle. This puzzle was used as the first attempt to find the simplest physical puzzle that jumbles. It is about the same radius as the unbandaged one (by Guilty Bystander) shown Figure 0. Jaap purchased one from me and then spent some time providing a solution for it. He was quite taken with the way it looks symmetrical but is not symmetrical. Jaap said it caused him a lot of extra effort to devise a nonstandard solution for it. It does not form a nice group like doctrinaire puzzles do.

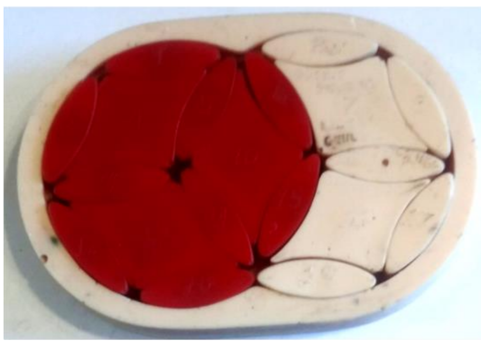
Figure 24.2 below shows 5 of my circle puzzle designs. The top left figure uses two colored trapezoids to make an Enigma type design that only needs two colors to uniquely color all the pieces. The original EE needed 6. Top right is similar, but a few pieces are duplicates. The 2^{nd} from top is my petal design 1981. I Sold a few only. It is very similar to the bottom right puzzle in Fig. 23.1 even to the size of it. The 2^{nd} top rt.



Simple Enigma puzzle. Only 2 colors needed, 1981.

Enigma puzzle. Only 2 colors needed, 2019.

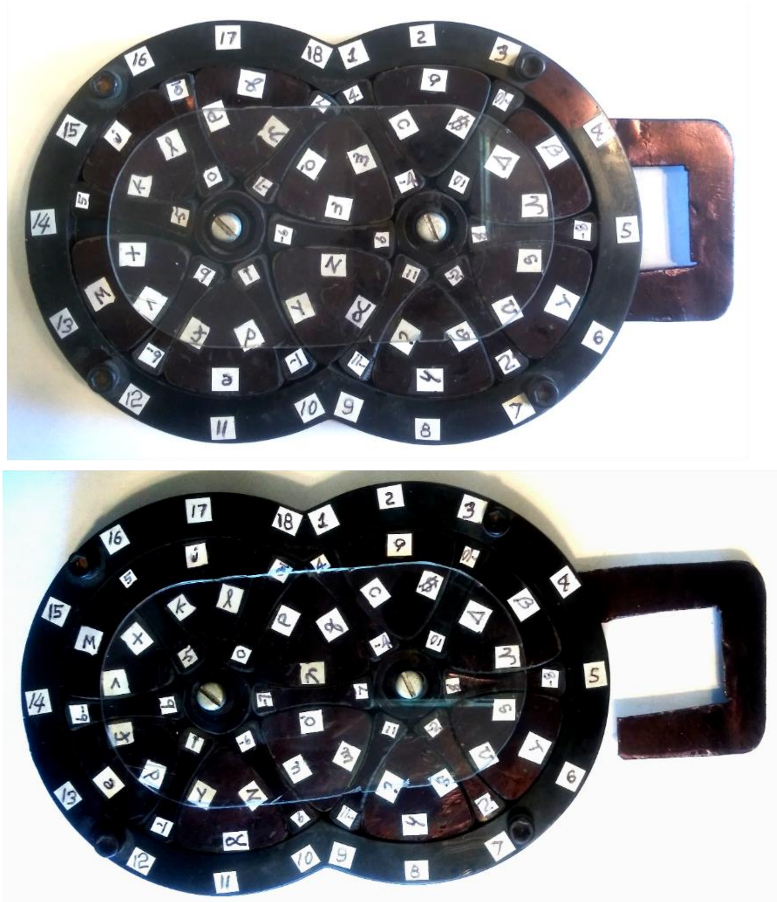
Wooden Engel's Enigma puzzle. 5 colors, 2016.



A French 13 piece circle puzzle, Rotascope, featured at the 6th IPP, 1983.
by Raoul Henriques Raba produced by Pentangle 1981. Bandaged.

Figure 24.2

is a wooden EE shown mixed up. Third top rt. Is a wooden Color Wheels solvable without memorizing an algorithm but one would help considerably. A simple zig zag movement puts all the bones in place.



Circle puzzle prototype combination lock coded as e-13, w-15, to unlock. 2 pin direct acting model 9-15-1981 D. Engel

Figure 24.3

This shows Figure 24.3 a coded combination lock circle puzzle. Only two pieces have to be correct to unlock it. It is the only existing prototype.

This kind of idea can be used to build code that would be difficult to crack. Think of it. You can make several dummy rotations before finally making the coded rotation for each letter you want to encode. A comma could indicate the final rotation. Someone without the starting setup would be pretty lost in trying to decode it.

Figure 25 shows 11 versions of my initial Slide Rule Duel series laser cut puzzles. Some of them were never sold but tested as prototype such as Yin Yang Yoga and Pie. These turn out to be too difficult or not sold for other reasons. Pie was deemed especially difficult but was interesting in having a kind of geometric basis. You could solve one or more of the polygons but

Figure 25

solving all three was a challenge. Heptalive was quickly discontinued as the pieces tend to be unstable due to the simple edge holding design. After a time Pentaplenty was also discontinued for similar reasons. Pie 3 and Pie 6 were and

still are very nice with Pie 3 being a bit too easy. Rotocross and Rocketeer are still very nice medium difficulty puzzles. Binary Bisect is kind of famous as a very simple but challenging bandaged puzzle. I am indebted to Jaap Scherphuis for mathematical analysis of some of my puzzles. This link also provides links to several of Jaap's analytics for some of my other circle puzzles, etc.
https://www.jaapsch.net/puzzles/battlegear.htm

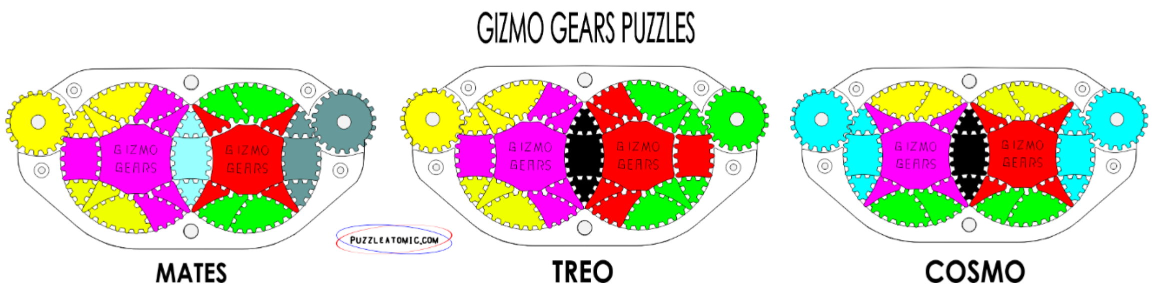
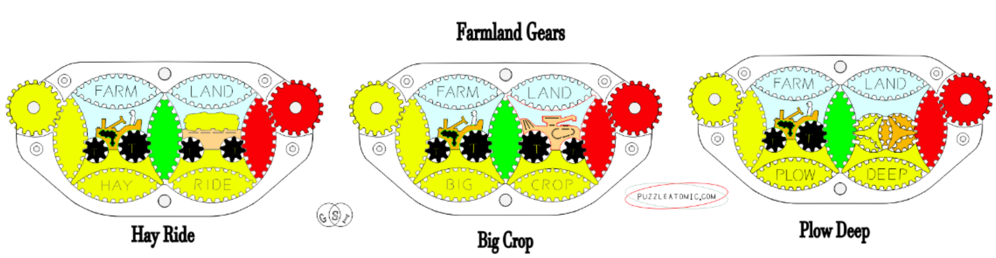


Figure 25.1

Figure 25.1 shows the Gizmo Gears sold in three versions. This was the puzzle that led to the discovery of the fractals after a lively and fascinating online discussion. Jaap did not analyze it, probably with good reason, as it is quite a bandaged puzzle. The three versions show (3,4), (3,3), and (4,4) symmetry even though all three have the same geometry arrangement. You could rotate it 60, 90, 120, 180, 240, 270, 300, and finally back home at 360 degrees. This rotation sequence exhibits both hexagonal and square symmetry which becomes dodecagonal when you try to unbandage it. It would be most interesting to see what kind of fractal it would produce if restricted just to the rotations it was physically designed for N=(3,4)

Farmland Gears



Farmland Gears are the simplest circle puzzles here. The cases,

Figure 25.2

borders and drive gears are identical to Gizmo and Battle Gear puzzles.

Slide Rules Deep Cut

After some time, mid 2010’s I designed slide rules where the two circles begin to cut into each other making for a very fascinating design. Oskar has used this idea for his Weird Disk 4xinfinity design which I like. The slide rules were never patented but not for not trying. The patent examiner said he thought they were obvious if

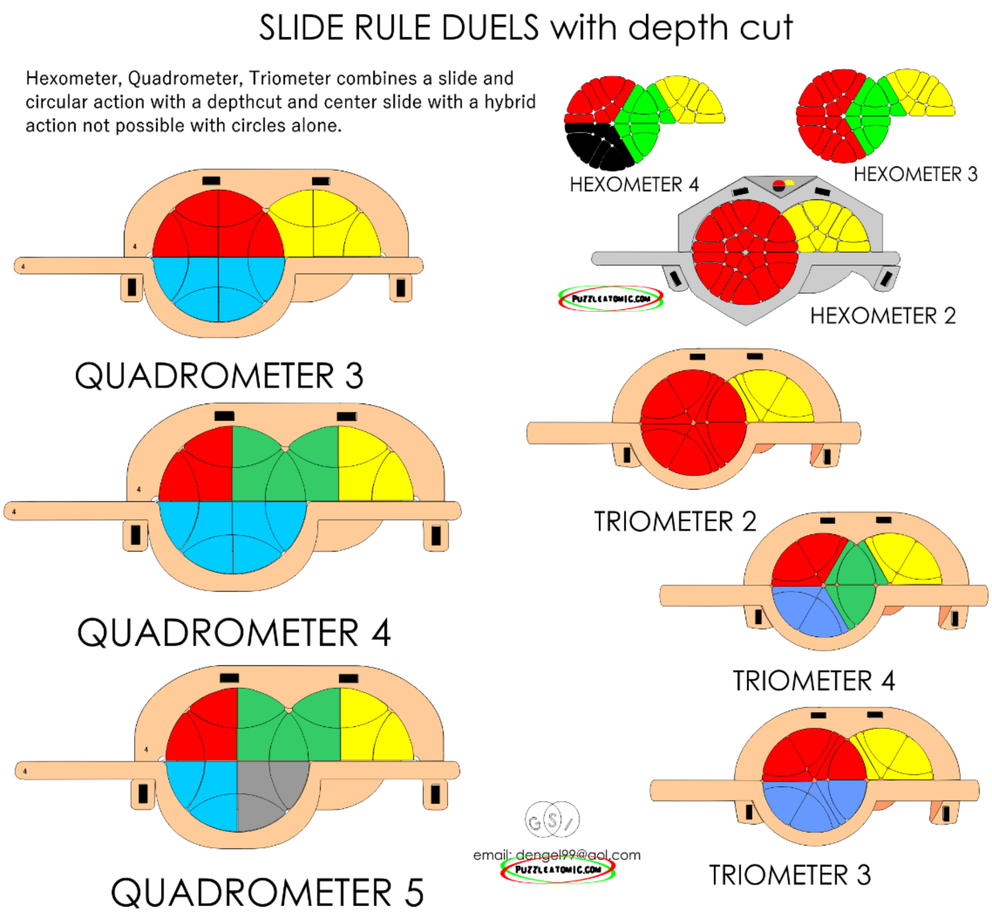


Figure 26

my current designs were known by the art. Too me this was very strange indeed! Figure 26 shows these deep cut versions. The Quadrometer is my favorite because I can solve it without a lot of analysis.

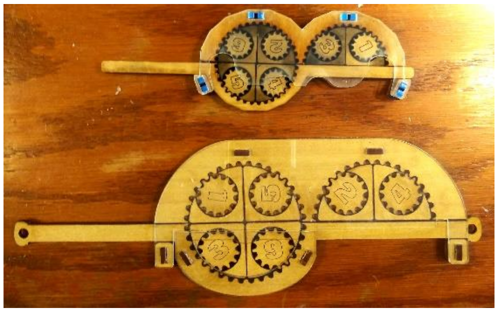


Figure 26.1

Here is a prototype, called Planets. To make it less frustrating the gear teeth numbers would need to be changed. As it is the large circles have 36 teeth while the planets have 14 teeth. Seven and 18 are mutually prime preventing a quick solution.

Slide Circle puzzles

After years of circle puzzle design, I tried a cut out on both sides of the encased pieces (Figure 27). By having a cutout on both front and back, your thumb and finger could grip a piece and turn a circle much more easily than by pushing a piece from one side only. This idea was used long before I thought of it. The simplest ideas are often mentally blocked out of our minds perhaps by not systematically considering every possibility. The puzzles shown here were only prototypes, but I probably should have redesigned all my gear puzzles as non gear this way. The problem with this is that your fingers get sticky oils on the edges causing the puzzle to become harder to turn. With the gears, your fingers never touch the pieces so they stay perfectly clean and you can hand color them without smearing the colors. My patent on the gears is still in effect.

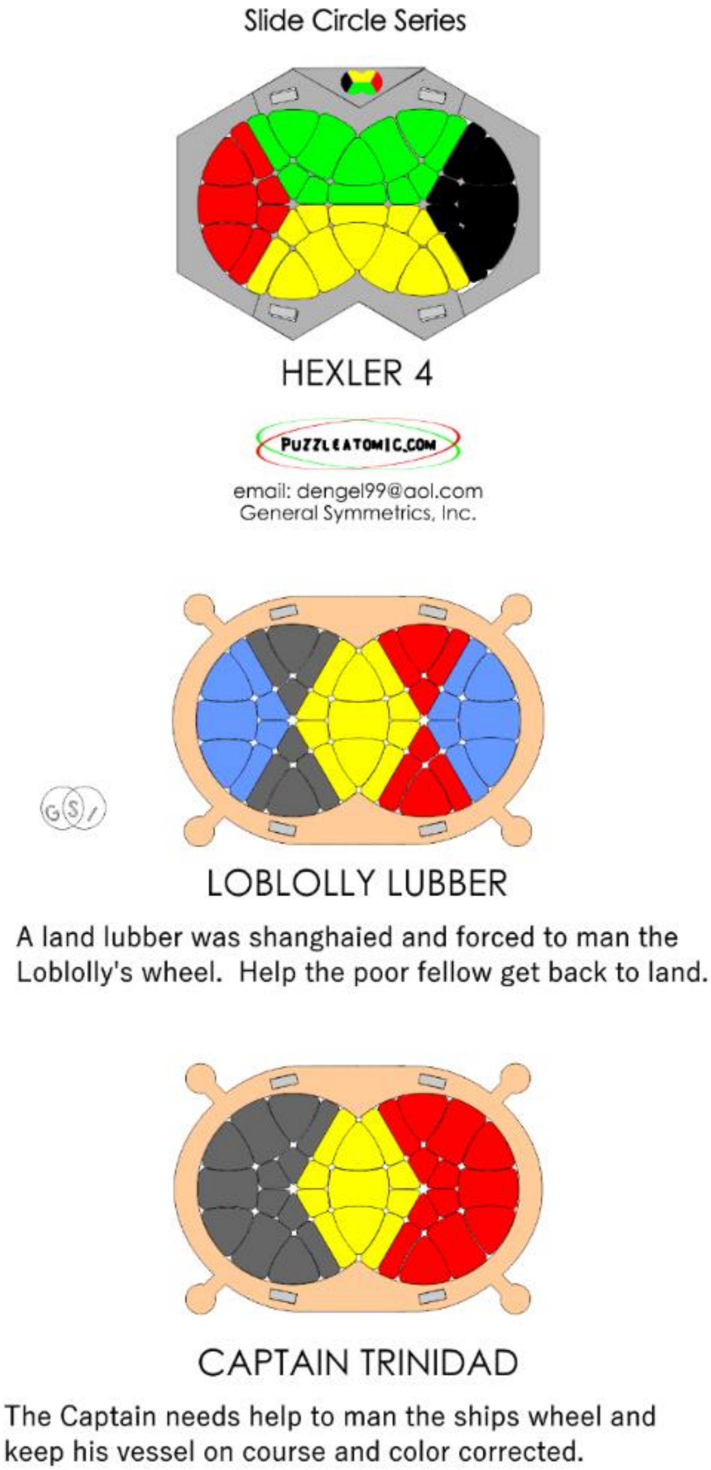
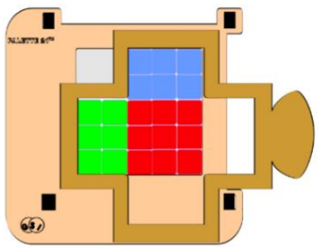


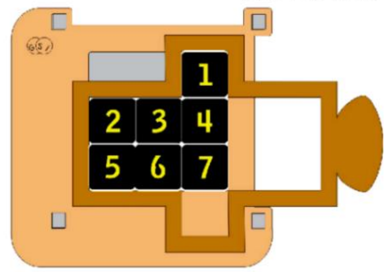
Figure 27

Palette Puzzles

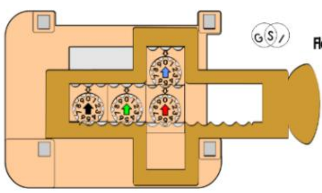


PALETTE 21
MIX UP THE PIECES FIRST
1. GET A 2X2 OF ONE COLOR 2. GET A 2X3 OF BLUE OR GREEN
3. GET A 3X3 OF RED 4. SOLVE THE ORIGINAL PATTERN

PALETTE 7



Palette 7 looks simple but you will find it to be quite challenging.

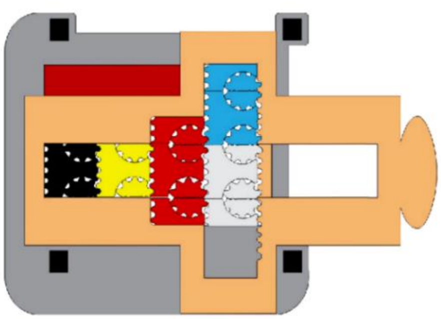


Palette Series
Flatland Gear Works Puzzles
CARTESIAN PLEASURES
DIVISION

PALETTE MIX 4

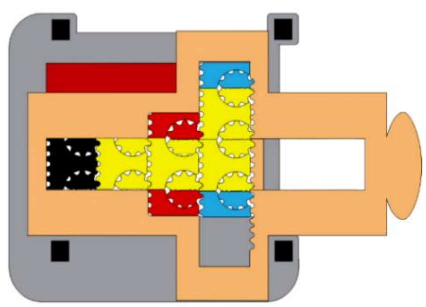
Many number cycles are possible. A cycle of 30 can be generated by moving in a full 3,1,3,1,... clockwise motion. There are 330 of these cycles of 30 and 10 cycles of 10, so that 30x330= 9900, and 10x10 = 100, 9900 + 100 = 10000. Thus 1 % of the cycles are degenerate. Find a degenerate cycle. Try to find moves that will generate the counting numbers, or the squares, etc. Can all numbers 0000 to 9999 can be generated?

PALETTE CUT 2 TM



The Palette Cut 2 is suggested for the enjoyment of the arts. Watch the little gears race around and form new alliances. Not as difficult as it looks but more challenging than Palette Cut 1.

PALETTE CUT 1 TM



An artist can use his spatula to cut and and mix his paints. In this spirit the Palette Cut 1 is presented for the enjoyment of the arts without messy paints. Enjoy the colorful mixing action and changing colors. Not nearly as difficult as it looks.

Figure 28

The Palette puzzles are not circle puzzles. These puzzles slide square or rectangular pieces in straight lines. Challenging 'twisty' puzzles, each palette move leaves all the pieces in contact and uses only straight-line moves. Use a handle to slide the pieces vertically and horizontally. By having some pieces cut in half, etc. they can be designed as bandaged. I made several prototype palettes. Figure 28 shows 4 designs, Palette 21 with 21 pieces, three colors, Palette 7 with 7 pieces numbered 1 thru 7, Palette Mix 4 with four 10-tooth gear pieces, and Palette Cut 1 with 4 colors and Palette Cut 2 with 5 colors. I believe the Palette Cut was the puzzle that started the 3D geared puzzles first brought out by Oskar van Deventer. Palette Cut has gears that are split so that individual gear pieces can be rearranged. The handles can also be varied in all sorts of ways.

Figure 29 shows six more Palette designs. Palette Eye is a bandaged puzzle with 14 1x2 pieces and 1 2x2 piece in the center with a circular gear representing an eye in the center of the puzzle. The color pattern and eye orientation must both be correct for a solution. Palette coral uses colored beads and numbers. Palette Green has a triangular gear and six pieces to be arranged. It was presented at IPP28 in Prague.

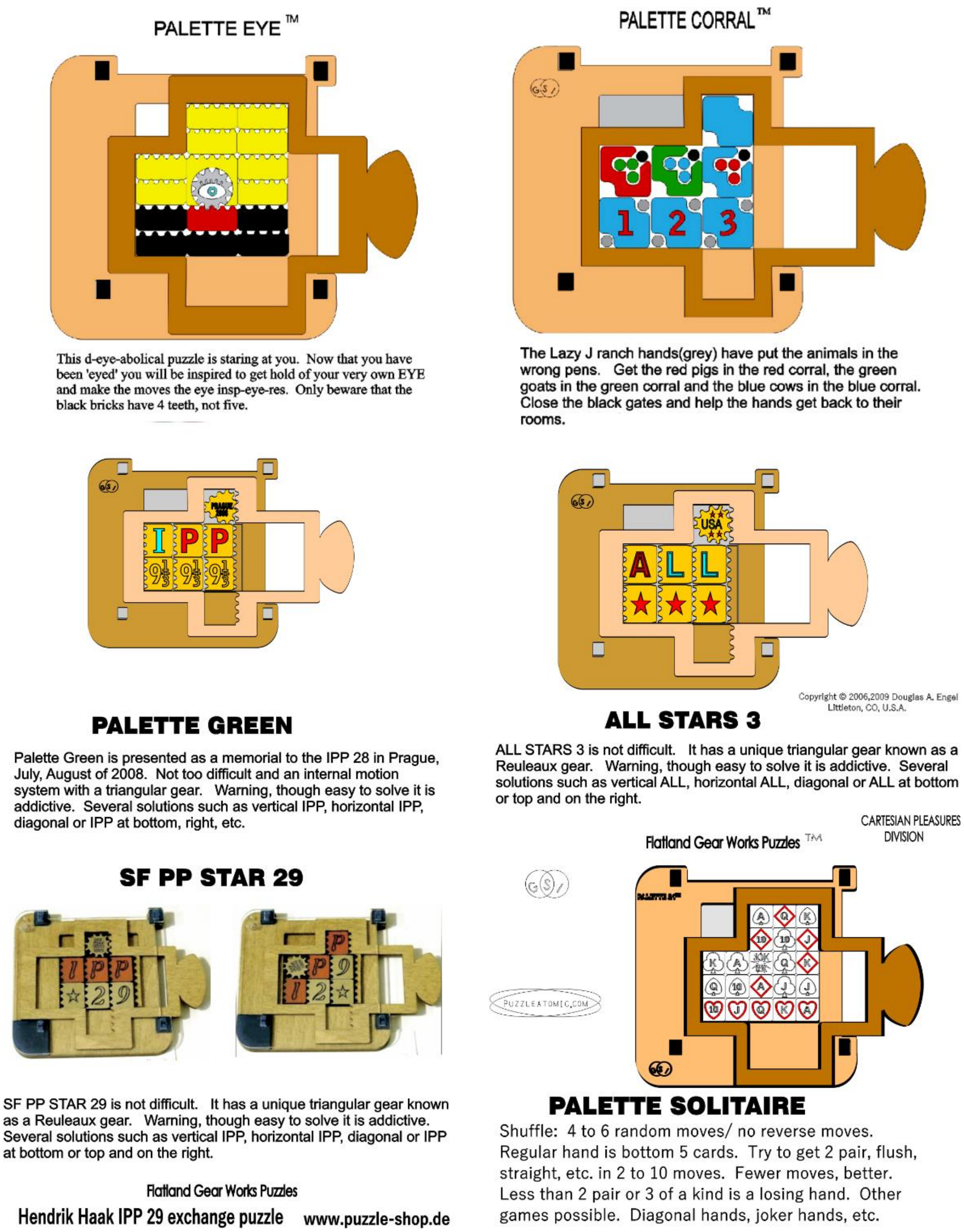


Figure 29

All Stars 3 Is just a redesign of Palette Green. SF IPP Star29, another redesign of Palette Green, was presented by Hendrik Haak, owner of Puzzle-shop.de, at the San Francisco IPP29 in 2009. Palette Solitaire was never sold.

Flatland

Many people designed circle puzzles over the years as beautiful injection molded puzzles, more in the spirit of twisty 3D puzzles, with no visible outer borders needed to hold the pieces in place. Thus, it was appropriate that A. K. Dewdeney presented my Engel's Enigma (which he named) in a Scientific American article discussing its merits and demerits. Dewdeney was impressed by the difficulty of EE but also was convinced that it had no 3D properties. It seems that not everyone agreed. Since the triangular pieces known as stones can change their orientation, they operate the same as corner cubies in a Rubik's cube. With three circles the 'bone' pieces can change orientation as well. The difficulty of solution is a con. However, this can be turned into a pro by using simpler color patterns. Thus, one puzzle can have varieties with a slope of difficulty, EE being the most difficult. In addition, the two center pieces can be embossed with an arrow, etc. so that they also need to be oriented correctly, further increasing difficulty. The center pieces of a 3x3x3 Rubik's cube could also have an arrow but there is no preferred symmetry to point the arrow to.

One purpose here is to point out that most of my flat twisty puzzles have flatland properties. If the borders could be secured somehow in flatland and if the borders had the property of flatland transparency so that flatlanders could peer inside to see the color locations of the pieces then a flatlander should be able to rotate the gears or push at the pieces through an opening in the borders to rotate the circles and mix up the pieces. The same reasoning also applies to my Palette series puzzles and to my Slide Rule Duel puzzles. Thus Dewdeney, who published a book called 'The Planiverse' was motivated to publish the EE. His book was a compilation of devices that could be built by flatland engineers. In fact the fractals discovered by Bob Hearn and others prove that the circle puzzles have a rich structure that will continue to enhance the progress with the twisty adventure started by Rubik. The inventive inspiration started by Rubik is our forever expanding treasure chest. Martin Gardner would have loved it.

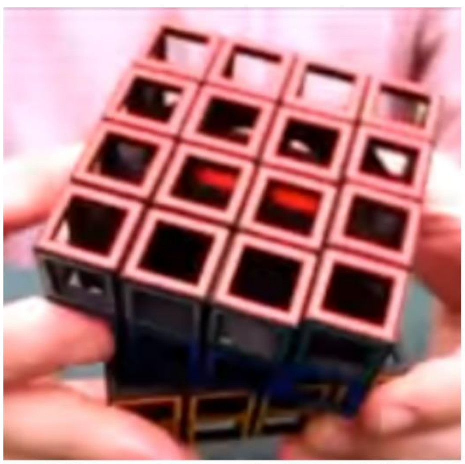
A property of flatland puzzles

One property owned by flatland puzzles but not by 3D twistys is that the positions of all the internal pieces are always visible. With a 4x4x4 cube you cannot see the pieces of the enclosed 2x2x2 cube. Of course, with a graphic puzzle you could see this but with a physical puzzle it is not so easy. One possibility is to put a 2x2x2 inside of a hollow cube. A hollow 3x3x3 is on the market. We want the pieces of the 2x2x2 to move whenever the inner slices of the 4x4x4 are moved so that a true 3D doctrinaire twisty results. Most 3D twisty puzzles are 2D spheroid puzzles, topologically two dimensional. In Figure 30 we depict a 2x2x2 inside a 4x4x4 inside a 6x6x6 cube. The inner pieces can be co-mixed providing a challenging, triply deep cut, 3D situation. The right-side image shows the two smaller cubes tumbling out. Will this ever exist as a physical puzzle? One simple solution would be to use magnets. Then you could lift off layers to see the solution progress of the inner layers. With translucent colors of the two outer puzzles this could be quite nice. Several magnetic twisty puzzles exist and new ones are brought out from time to time. This company uses magnetic dice: Magneticcube.com

Figure 30 Triple deep cut puzzle using magnets as the mechanism.

The 2x2x2 two color dice cube shown in Figure 30 is always two symmetrical arrangements of the two sets of black and yellow colored cubies (ignoring the dots), no matter how it is mixed up, an old discovery of mine. Wikipedia has a good article about the history of Rubik's cube:
https://en.wikipedia.org/wiki/Rubik%27s_Cube

Hold the phone. It turns out that Oskar has already created a 2x2x2 inside a 4x4x4 cube he calls the Framed Cube. Figure 31 shows Oskar demoing this in a video. Oskar used a clever method, opting to make the outer cubies as frames that you can see through, solving the problem of seeing the 2x2x2 in the center. I would love to see how this was done mechanically. Framed Cube probably incorporates some type of inner frame that holds the 4x4x4 cubies in place while also keeping the 2x2x2 cube centered against the inner faces of the 4x4x4. The magnetic idea with translucent colors on the outer cube would look like a mysterious crystal. It might also be too difficult to get enough clarity to make out the 2x2x2.



Oskar van Deventer's Framed-Cube

David Pitcher's Insanity Cube

Figure 31

The right most Figure 31 shows David Pitcher's Insanity Cube. Tom Cutrofello mentioned this to me. He said it lives up to its name. It seems appropriate to include with Oskar's Framed Cube, both being very eclectic and mind bending.

Set theory of circle puzzles

It is possible to specify the movements of elements of sets into other sets where the elements are the pieces of a twisty circle puzzle. This is familiar as a Venn diagram but would not have any advantage over the group theory method used by mathematicians. The idea is to find a 'set' mathematical description as another method of teaching group theory and set theory and solving twisty puzzles. There may already exist such a set-group description. Venn diagrams can be very complicated and sophisticated, but I have never heard of a Venn fractal. This would be a kind of branch concept, not true set theory since Venn diagrams are not meant to rotate their intersections out of one circle or area and into another. How would you write set equations for this?

This brings up another interesting thought. Can a jumbling puzzle always be solved? Suppose you have mixed it up making a finite number of new cuts. Hand it to a friend and see if the friend will be able to solve it. Since there is no group or algorithmic way out, he may never be able to solve it. The number of combinations of the pieces or ways the pieces can be arranged might be huge. If he could make new cuts this leads to a trap, now having to solve new cut pieces. Quantum and information scientists would argue vigorously that it could always be solved if only a finite number of cuts had been made. Of course, by solved we mean restore some original simple multi-color piece pattern that existed before the cutting began. How do you know that a series of solving moves is going to lead to a solution or will it lead to a dead end requiring reverse moves to back out or will it lead to an 'almost solution', then more moves trying for a final solution which just end up mixing it up again. Perhaps a computer could solve it, using trial and error moves, in a reasonable time.

What is the meaning of all this twisty stuff?

Twisty puzzles are useful as recreational mathematics and as teaching devices. They are also works of art, and valuable to collect. Twisty puzzles are fun to solve and are used in speed competitions. The field of twisty puzzles seems to expand every year and is a kind of puzzle technology of its own. These puzzles have taught us about crystallography, and how a puzzle can have some importance in pure mathematics and mechanics.

As mathematics twisty puzzles belong to group theory but as mechanics they belong to innovation. Will there ever be a practical use? Well all the above can be thought of as practical. Perhaps these complex entangled beautiful works of crystal puzzle art are nothing but adventures in thought. But will there ever be a use such as a kitchen item used to mix or grind, or a calculation device, or a device that is used to perform medical research? One practical use has already been mentioned. You could use a twisty puzzle as an encrypting device. I believe this was mentioned in my original US patent, 4415158 issued back in the 80's. A mathematician sees the entire field as pared down to the simple rules of group theory. From these rules, boiled down with great care, the entire field of group theory explodes. Wikipedia mentions the manipulations of Rubik's Cube as the Rubik Cube group. Here is the definition given by Wikipedia of what it takes to have a group:

A group is a set, *G*, together with an operation • (called the *group law* of *G*) that combines any two elements *a* and *b* to form another element, denoted *a* • *b* or *ab*. To qualify as a group, the set and operation, (*G*, •), must satisfy four requirements known as the *group axioms*:[5]

Closure

For all *a*, *b* in *G*, the result of the operation, *a* • *b*, is also in *G*.[b]

Associativity

For all *a*, *b* and *c* in *G*, (*a* • *b*) • *c* = *a* • (*b* • *c*).

Identity element

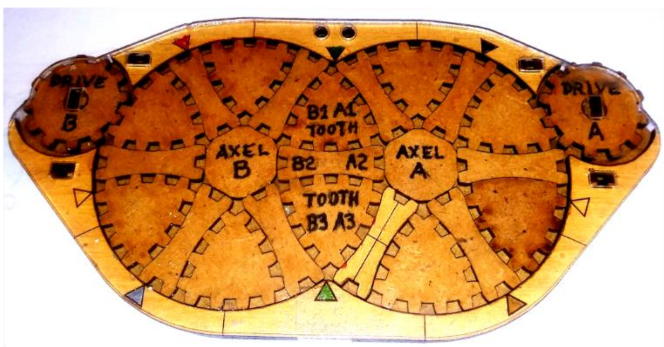
There exists an element *e* in *G* such that, for every element *a* in *G*, the equation *e* • *a* = *a* • *e* = *a* holds. Such an element is unique (see below), and thus one speaks of *the* identity element.

Inverse element

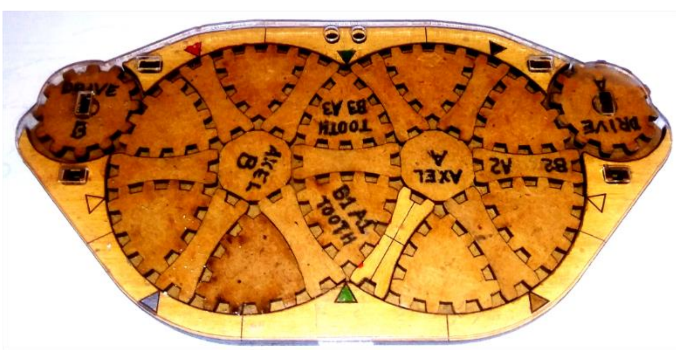
For each *a* in *G*, there exists an element *b* in *G*, commonly denoted a^{-1} (or −*a*, if the operation is denoted "+"), such that *a* • *b* = *b* • *a* = *e*, where *e* is the identity element.

(by generous permission of Wikipedia)

Twisty puzzles as gears



Home position. The pieces of a circle puzzle can be thought of as entangled gear teeth. The geared teeth and drives add an additional gear theme.



After a few turns the entangled teeth, axels and drives are reoriented.

Figure 32 here shows the idea of thinking of a twisty puzzles' pieces as entangled gear teeth. The pieces or cubies of a Rubik cube can be thought of as gear teeth. Rotating a face rotates all the teeth of that face. Then rotating an adjacent face rotates the nine gear teeth(cubies) of that face. Rotating a center section rotates the teeth of the center section. Since the faces can move independently but are entangled with adjacent faces you have a twisty puzzle. Get the three center pieces correct on this circle puzzle, then axels and drives may not end up oriented correctly.

Figure 32

Bob Hearn says in his G4G 2014 Gizmo fractal talk, mentioned above, that some areas in one of his fractals look just like gear teeth! He says, correctly, it is just a coincidence. However the gear-like interaction of the pieces do have a tendency to create gear tooth looking areas as some of the fractals develop at various radii ranges.

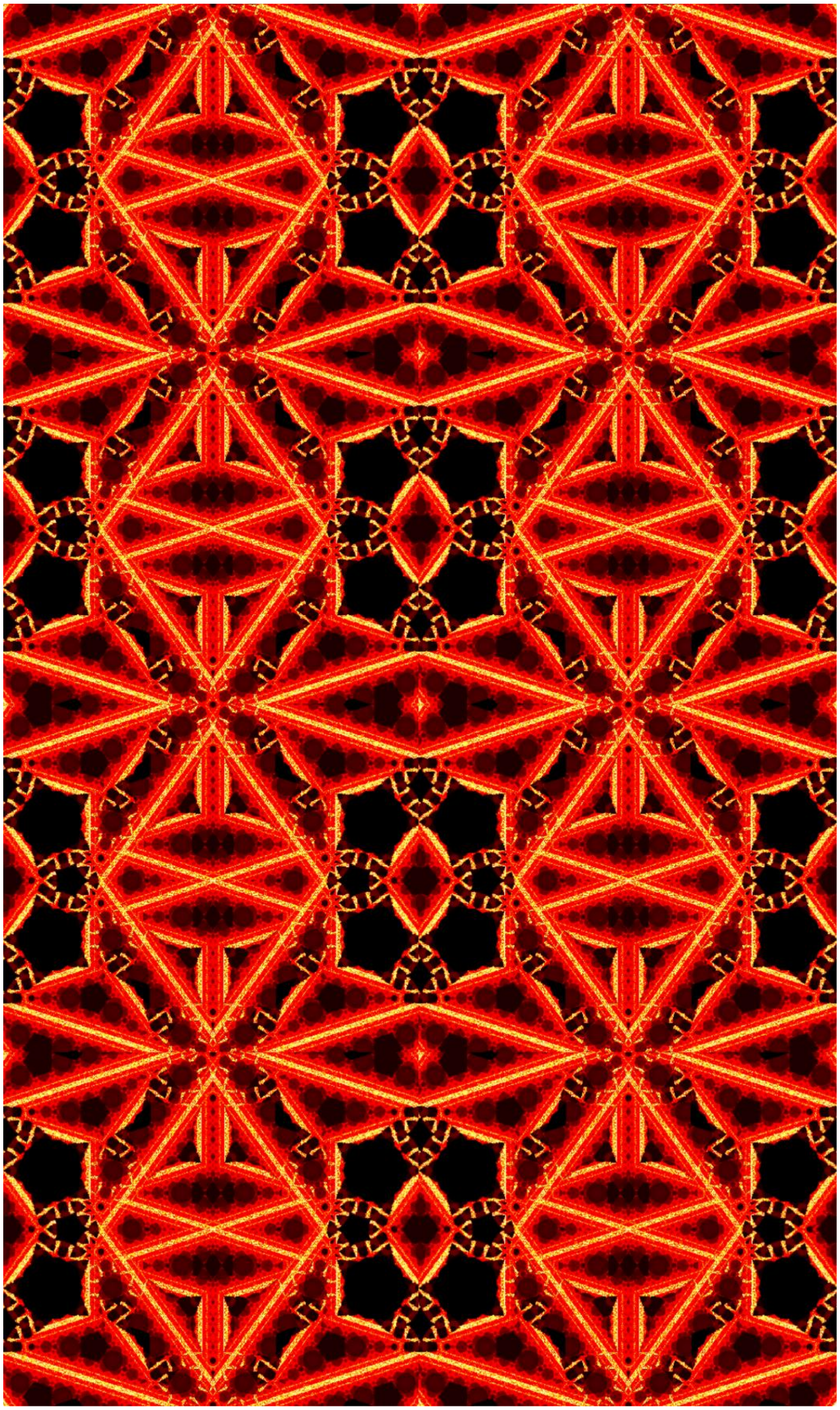
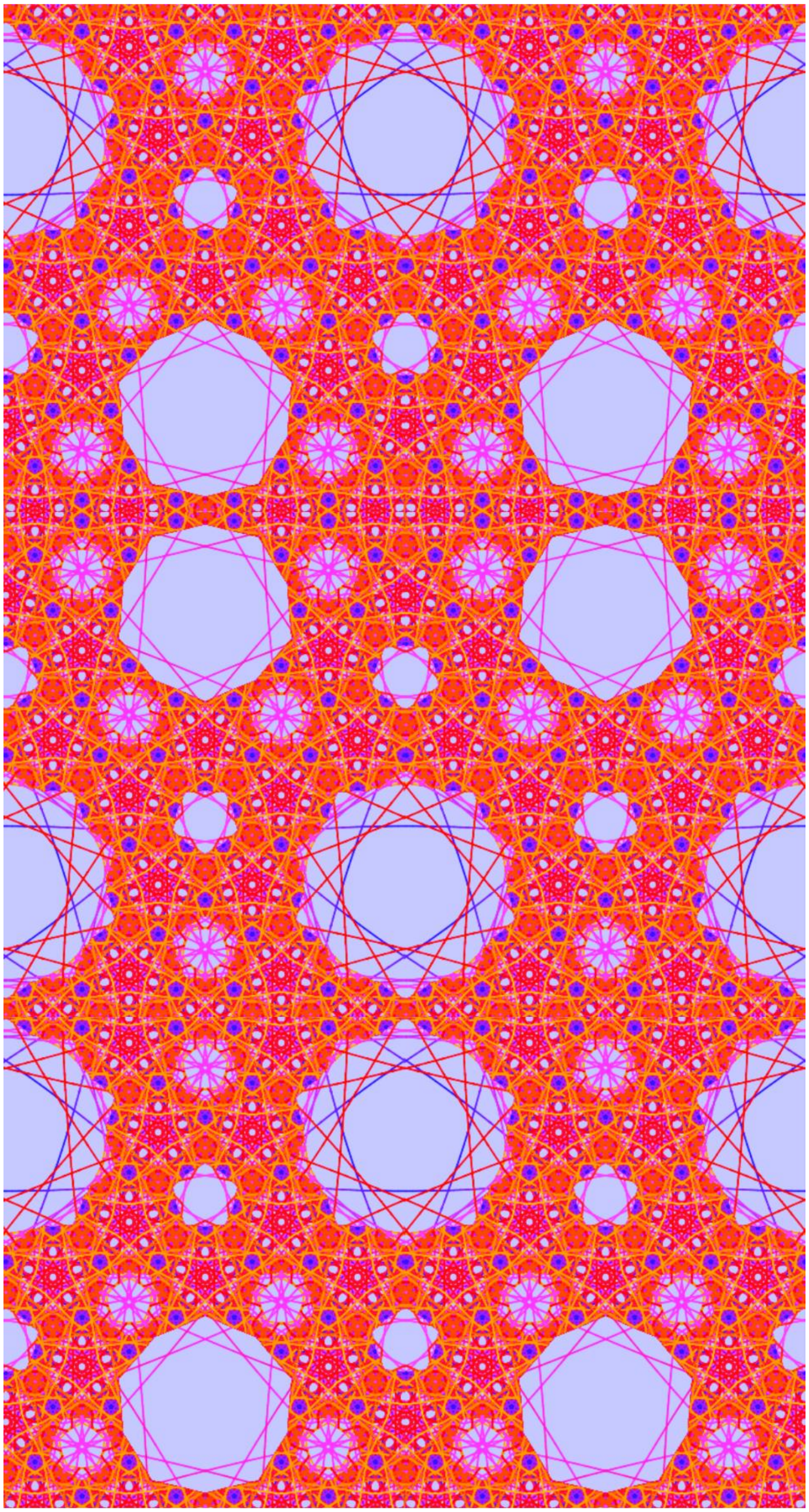


Figure 33

Fractal tiles for smaller N

Here is a tile derived from Jason Smith's Deluxe Render for N5, R2.69.

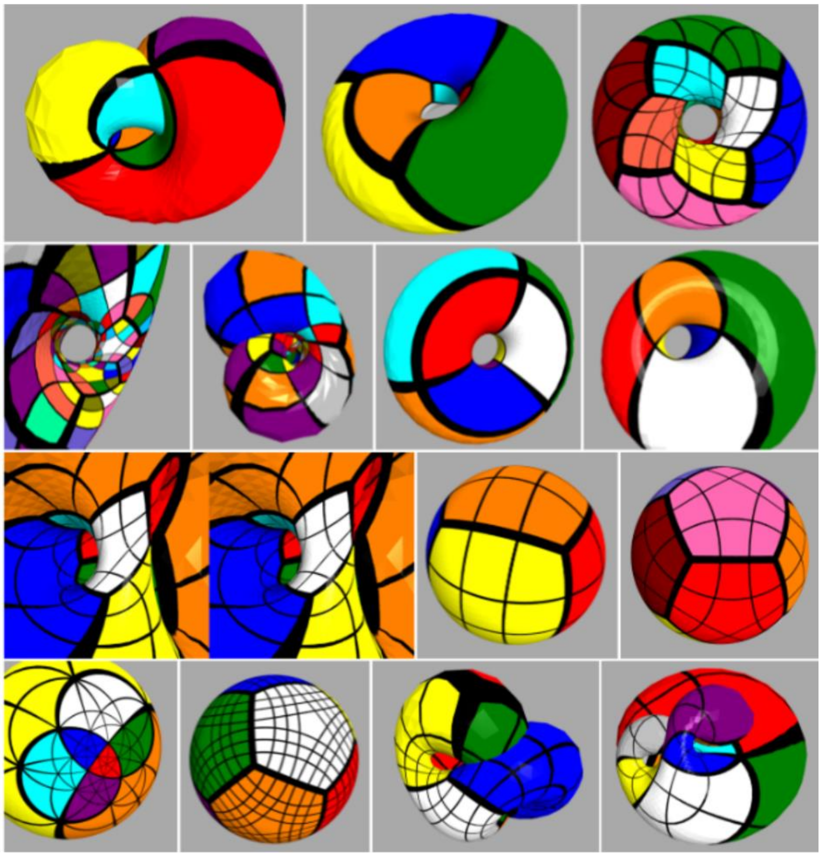
It was done as before by using mirror imaging to get rid of the asymmetry and clipping borders to make square edges. The final result has a pleasing symmetry. Of course in a real tile you might add and modify colors and could smooth the toothy areas, to make a professional tile design. It has an amazingly abstract and pleasing appearance: a crystal and fractal wonder.



One of Bob Hearn's N5 subcritical patterns from Deluxe Render

Here is a full piece cutting done with Bob Hearn's code for N=5 which I turned into a tile pattern. It is quite stunning. You can see that the full cutting code can produce wonderful graphics that are also works of art, pleasing to look at and would be great to use to tile a floor or wall or use as wallpaper. One could also see it as being used for carpet patterns, clothing and other textile uses.

Figure 34



Magictile program by Roice Nelson roice3.org/magictile/

High order code puzzles

Nelson Roice has written a cool program that presents a bunch of solvable circle type tiling's that have been projected onto various surfaces such as a sphere, torus, Klein bottle and so forth. It is available for free download at roice3.org/magictile.

These are a lot of fun to solve or to play around with and explore twisty hyperbolics.

Figure 35

Roice has used the idea of circle puzzles projected onto 3D objects to abstract the Rubik Cube. His article 'Abstracting Rubik's Cube' was published in Math Horizons copyright 2018 MAA. It has been included in the book 'The Best Writing on Mathematics 2019.

He can be followed at roice3.org or on Twitter at @roice713.

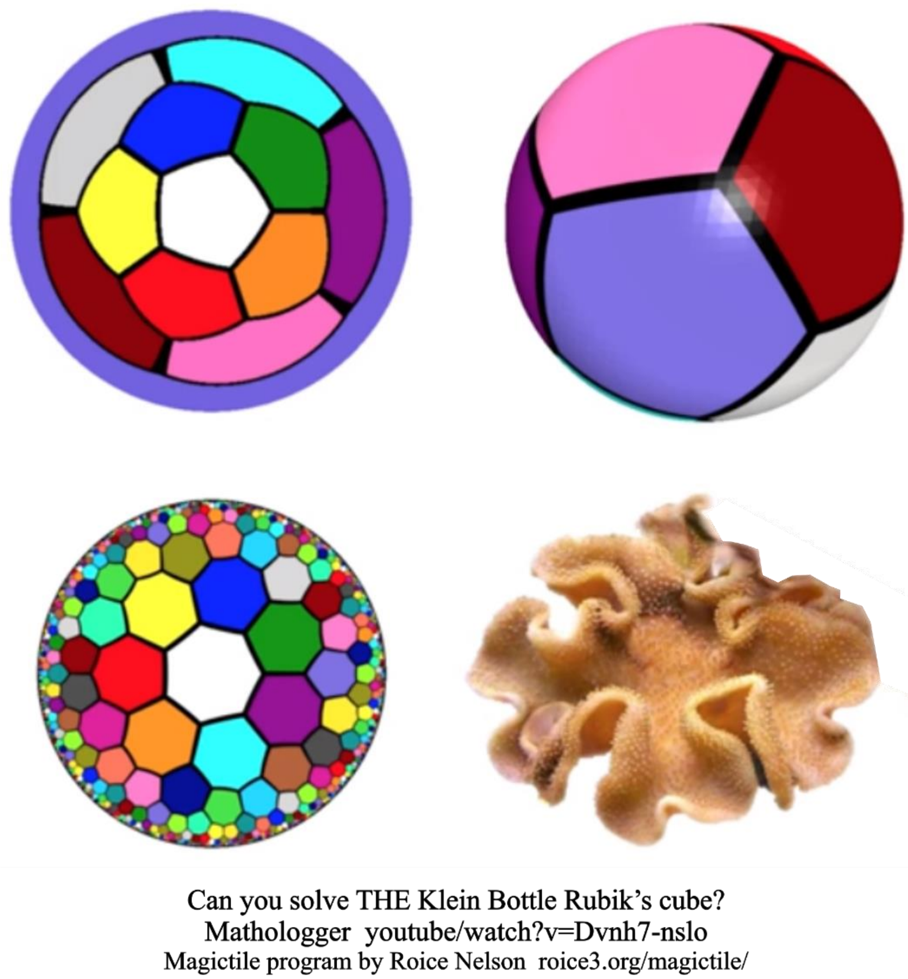


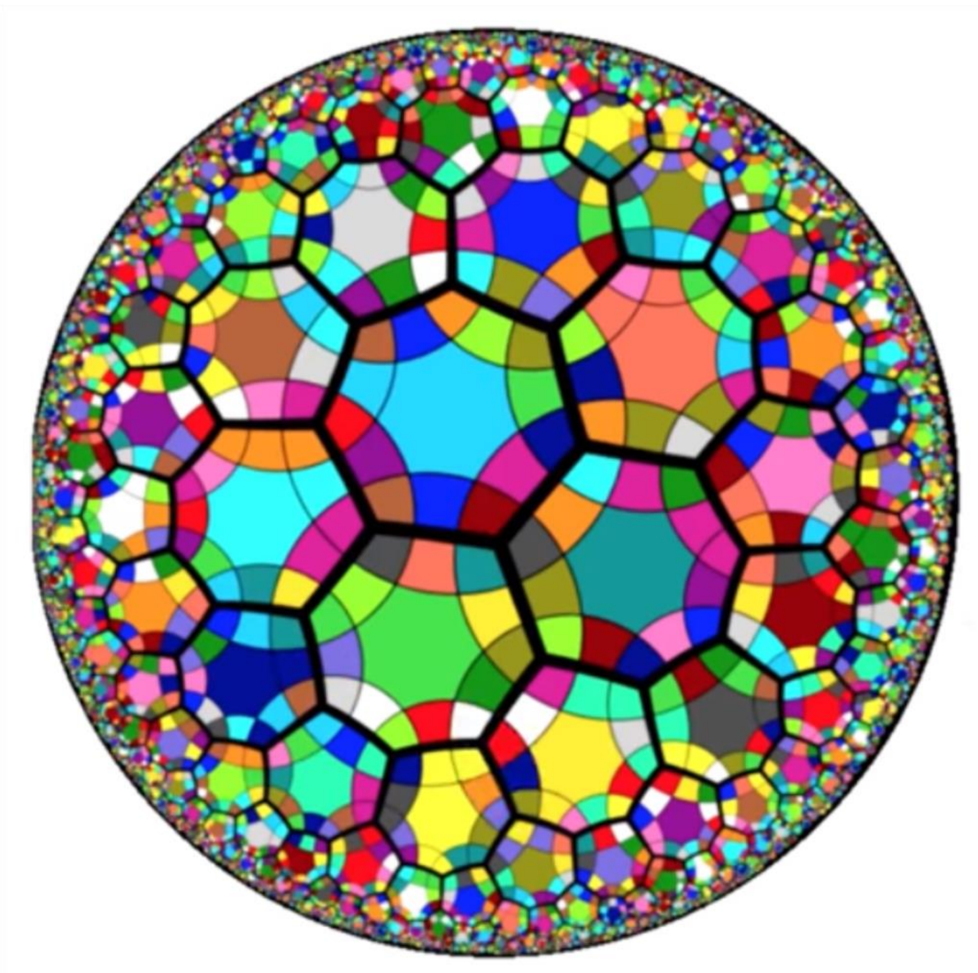
Figure 36

Figure 36 is a screenshot of Mathologger's youtube about Magictile. It shows a negatively curved form and how that idea is used to make perfect heptagon tilings on a spherical projection. As you rotate the sphere each heptagon can be shown to be a perfect heptagon. Two projections are shown with the 2 dodecahedrons being positively curved.

Can you solve THE Klein Bottle Rubik's cube?
Mathologger youtube/watch?v=Dvnh7-nslo
Hyperbolic heptagons circle puzzle
Magictile program by Roice Nelson roice3.org/magictile/

Figure 37

Here we we see Mathologer, Burkard Polster in Australia, showing a hyperbolic circle puzzle from the Magictile program by Roice Nelson. It is based on a small tiling patch of colored heptagons that is then repeated by infinite tiling. Turning one circle turns all the same repeated circles of the tiling so that you only need to solve one basic tiling patch to solve the entire infinite tiling. Mathologer has many videos about mathematics that clarify things in ways not taught in any book. His videos present both symbolic and visual reasons for various mathematical principles. So, for free, you can watch these videos and get a privileged view of many different mathematical ideas. In one video he shows why e to the i*pi equals minus one by a complex number kind of visual rotation when solving the power.



Can you solve THE Klein Bottle Rubik's cube?
Mathologger youtube/watch?v=Dvnh7-nslo
Hyperbolic heptagons circle puzzle
Magictile program by Roice Nelson roice3.org/magictile/

Figure 38

Here we see the heptagon circle puzzle randomized. It makes you wonder if at some small-scale nature is like this, totally random (mixed colors, and you assume that the circles are no longer in connected patches) and totally precise (perfect heptagons) at the same time.

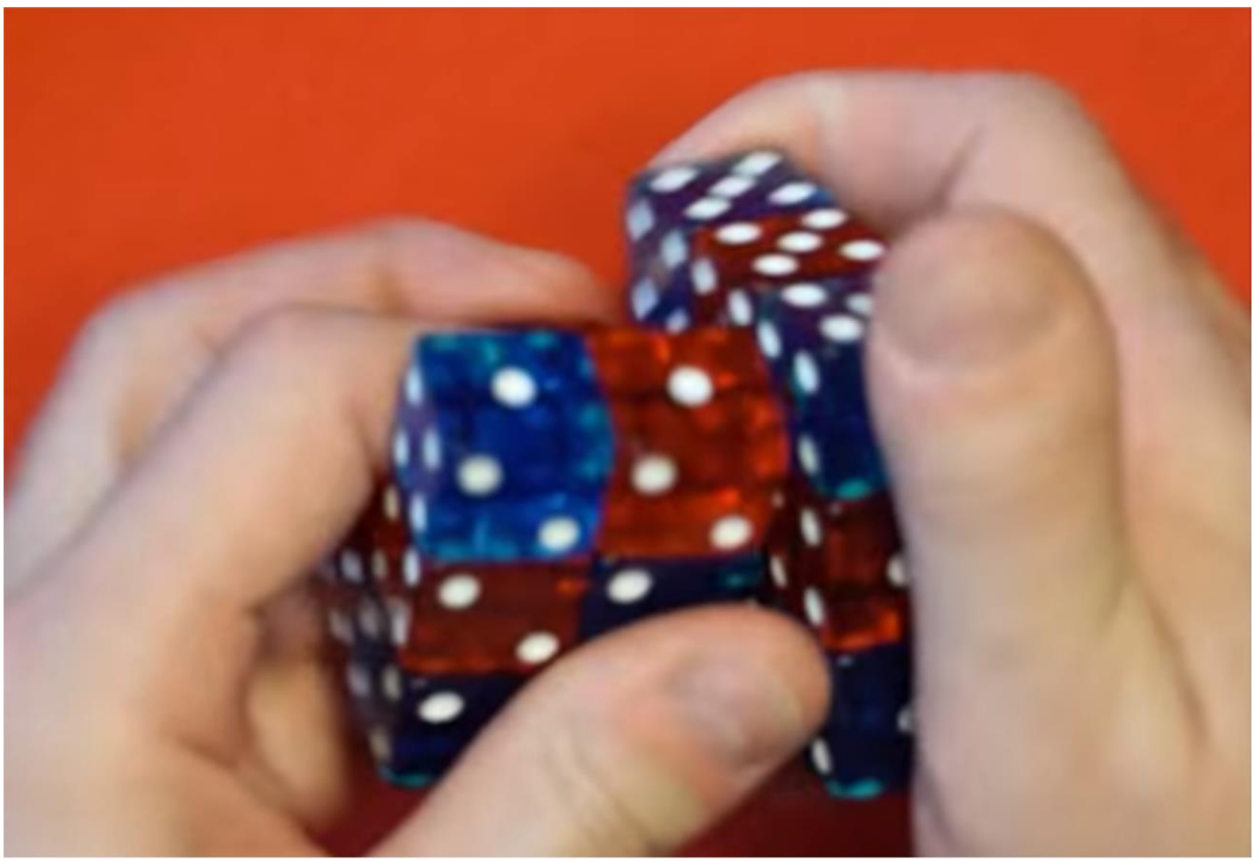
Can you solve THE Klein Bottle Rubik's cube?
Mathologger youtube/watch?v=Dvnh7-nslo
Magictile program by Roice Nelson roice3.org/magictile/

Figure 39

Figure 39 shows Harlequins, a circle puzzle using binary cuttings to simplify their solution. The large rotated green piece, near center, in the hyperbolic tiling can only have two positions, while the two small green pieces can commute around in the puzzle. Since the tile patches repeat it does not matter which circles you use to solve, since solving one patch solves the entire infinite tiling.

Mathologer demo of magnetic cubes

His is a screen shot of Mathologger with a Magneticcube.com magnet cube. It has the advantage that you can remove a 2x2x2 and just play with the 3 remaining 3x3 faces or play with the 2x2x2 cube.



Math Meets the Worlds largest Magnet Rubik's Cube Aug 28, 2015
youtube.com/watch?v=Xb8ENIS-5Go

Figure 40

Some quotes taken out of the fractal forum (with permission)

With over 250 posts on the Gizmo forum I have taken liberty to try to reprint those that seemed relevant to the theory of Gizmo jumbling as it developed. Probably someone else may have chosen differently but I think I have captured at least the powerful creative discussions that went on, with experts in math, physics, puzzles and computer science.

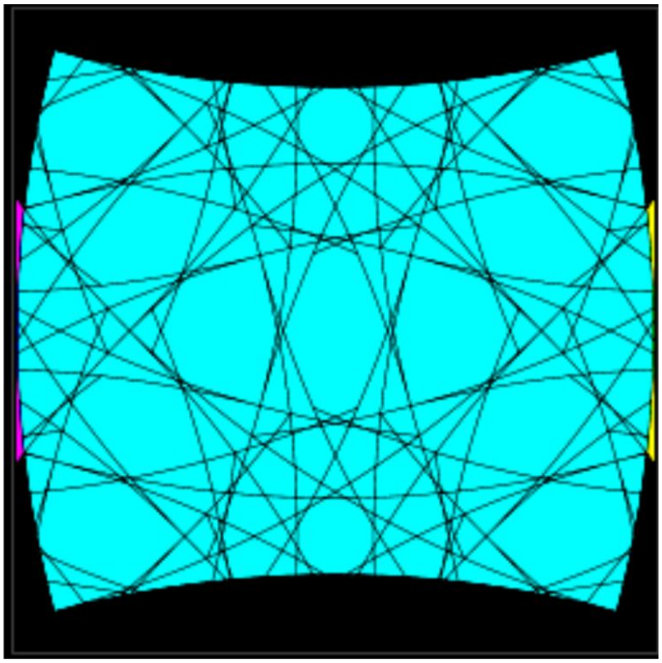
I have my own theory. The surd circles presented above show how an irrational N produces very precise bilateral or mirror symmetry plus precise counting up and then counting down of identical pieces. Also the pieces are large to begin with then get smaller and smaller and more numerous as the clock ticks away. This is a simple mathematical behavior due to the infinite symmetry of the circle and the periodic process of always doing the same rotation amount.

Taking this a bit further and applying to the Gizmo fractals we see that periodic rotations (using a simple symmetric generator such as [A', B]) alternate back and forth sending a given piece into circle A, then into the intersection, AB, then into circle B(if R is small enough). What we have here is periodicity of A going to periodicity of AB (since it always has to get back into AB in a periodic way) then going to periodicity of B and so on: AB, A, AB, B, AB, … . If R is larger a piece may stay permanently in AB. That implies a periodicity also and cutting of the confined pieces would not occur, thus a circle of confinement would be defined.

Now apply the surd idea (since N=5, 7 or greater is surdy since it is not a planar repeating tiling) and you can see that pieces will be cut in a strict and precise manner if the generator is repeated such as [A',B]. identical pieces will start with a one kind piece increase per cut, to a maximum then that piece kind will decrease one at a time to a minimum (as it gets cut up) and finally disappear by successively cutting each one of that kind. Of course, a single cut might cut thru several pieces but each piece in a multi-cutting will be a different kind of piece. I am not sure if they only get cut one at a time or if several might get cut at once. I think if the generator changes randomly then **cuts through several at a time** would occur.

Thus you could see jumbling disappear than reappear as R changes since the generator that does the cutting may see places of R where three or more circles cross node points simultaneously reducing the number of pieces at those critical spots of R. This was discussed by me below: … "As the radius increases at certain places if three or more arcs all cross at a single node point the piece count drops since those arcs no longer make separate cuts when they come together at a single point. This means that if the number of pieces is large the piece count can drop dramatically. From this there could be a possibility that jumbling could stop(for a specific generator) at single R sizes when getting to such an arc meeting spot (considering symmetry such crossings will happen at several places simultaneously). This is also discussed above in Fig. 5.1. …" DE

Page 1 Wednesday July 10, 2013 Thursday, August 8, 2013



Carl Hoff page 1 showing over 5 days 9 hours of iteration.
http://twistypuzzles.com/forum/viewtopic.php?f=1&t=25752

"I'm pretty sure they don't jumble, but I can't prove it. They look like a bandaged version of a puzzle with 12 states per wheel." Jared, page 1

"I should be able to take a shot at this over the weekend. The proof should look just like a more complicated version of GuiltyBystander's unbandaged Battles Gears. So I'm rather certain Gizmo Gears doesn't jumble in principle. In practice I suspect we'll get pieces on the order of (or maybe even smaller) than the size of the gear teeth so if you were to cut up an actually Gizmo Gears puzzle you'd end up with a (physical) puzzle that no longer functioned." Carl Hoff, page 1

"This gets increasingly complicated. I can't wait to see where this leads us to." Andreas Nortmann, page 1

"I'm starting to see symmetry requirements that I don't think I'll be able to satisfy with a finite number of cuts. I'm not sure how to prove that short of making an infinite number of renders and I doubt I'll be able to make many more iterations going at this with POV-Ray. But I'm now almost convinced that this puzzle jumbles. …" Carl Hoff, page 1

"After discussion with Carl and Oskar at IPP, I've written a program to investigate unbandaging for this general family of puzzles, parametrized by disk radius and number of turn states.

I don't have an answer yet for N = 12, R = sqrt(2) (Gizmo Gears, I think... does (at) R = sqrt(2)?). I suspect based on behavior for other parameters that all such puzzles can be unbandaged. However, the number of pieces for Gizmo Gears would be astronomical, on the order of hundreds of thousands, at least. I ran it up to 100,000 cut iterations without bottoming out. I am working on a second version of the program, which should give exact answers, but this is much more complicated, and may take me a while." Bob Hearn, page 1

"So they don't jumble, but they're realistically impossible to unbandage? Awesome!", Jared, page 1

"It looks like it should be straightforward to prove that as the radius expands every time there's a crossing the number of pieces remains finite. Proving that the number of such crossings which happen is finite I don't see an obvious approach to." Bram, page 1

"…Really, I think this is a very fascinating question! I think this should be a problem of more general mathematical interest, outside of the twisty community.

This is what N = 7, R = 1.7 looks like, zoomed in 100x, at 250,000 cut iterations. The image hasn't changed much as I've zoomed in and cranked up the iterations and the sampling resolution." (image no longer displays on the forum)DE Bob Hearn, page 1

"If any of these are in fact jumbling, there's the very interesting question of for any given N, what's the smallest value of r for which the number of pieces is infinite?" Bram Cohen, page 1 (first suggestion of critical radius)

"I was pretty sure that no flat, two-circle puzzle would jumble when N is an integer. This result is troubling. I don't understand why it jumbles. …" Brandon Enright, page one

*"Geeze, people expect me to be the oracle of Delphi or something.
I have no idea whether this puzzle actually jumbles. My initial guess was that it does, but I was wrong in guessing/assuming that Battle Gears jumbles, so I'm declining to take any bets.
There is at least one puzzle which has only rational angles but does jumble. It's the one with three axes at 120 degrees to each other, each of which can rotate 90 degrees, where the actual build is a little bit fudged. I'm unfortunately spacing on the name. There was an earlier thread on that one, and the pattern of there being an infinite number of pieces is quite simple and understandable. It isn't a dramatic explosion of pieces at each iteration like these appear to be though." Bram Cohen, page 1*

"…Back on topic, my brain fails to comprehend how a rational angle turning puzzle can jumble. (Does it mean that any puzzle that has irrational turning angles jumbles?)" Rubilcollector123, page 1

Comment by Doug Engel: My original CPM discusses the idea of an infinite black circle where if R is huge the number and kinds of pieces increase without limit. Circle puzzles incorporate the circle, a kind of fractal as it is a polygon with an infinity of edges. Pi is transcendental. Also Lie algebras are based on the infinite symmetry of the circle (all rotations leave it unchanged). Waves are expanding circles and form more complex interference patterns when waves from 3 or more sources interfere. Thus, circle puzzles should produce all sorts of cool geometries that may or may not occur in nature. Their use in art and math is has value.

"Stop the presses!!! I have an answer. 😀 **N = 7, R = 1.7 definitely jumbles.**

This is an image of the rightmost point of the left disk, transformed under all possible rotation sequences, stopped at 5,000,000 distinct points. Shown at 10x relative to my earlier images of the complete puzzle.

So how exactly do I know it jumbles? Let me back up and describe what I've been doing. It started with the observation that the complete set of unbandaged cuts is just the image of the two arcs bounding the disk intersection, under all possible rotation sequences. If this is not obvious, then imagine an unbandaged puzzle, and look at any cut: that cut must have been generated by some particular turn that chopped a larger piece at some point. But that means that undoing some rotation sequence must put it back onto one of those primary arcs, otherwise it couldn't have been cut. …

(images not available on the forum)DE

…I realized -- aha! I don't have to store all the cuts. If a configuration jumbles, then the image of some particular **point** on a bounding arc must itself have an infinite image under all rotations. If all bounding points have finite images, then so does the entire arc. …

… For a given point, just record where it is, and record whether we have explored (left, right) x (clockwise, couterclockwise) from that point. Keep exploring existing points until all moves have been explored. Then, when we get to the next sampled point along the arc, we can clear the hash table. So we don't have to store all the cuts, just those for a given point. …

…boom! Right off the bat, N = 7, theta = 0 (rightmost point of left disk) generates ~~an uncountable,~~ a seemingly infinite set right at R = 1.7 (and not at anything <= 1.69). True, I've only iterated to 10,000,000 distinct points. But there is an apparent complex, perhaps fractal pattern. …

…It's clear in hindsight what happens at a critical radius that enables the transition to jumbling. If the jumbling generator sequence is e.g. RLLrrl, then it's only when a source point stays within the disk intersection under that sequence that we can jumble, and the intersection grows with increasing radius. (We can always ignore moves that take points outside the disk intersection, because they must re-enter the disk for anything interesting to happen to them and circling all the way around is equivalent to backing up within the intersection.) **It's also conceivable that there is no single generator sequence that does it, but a set of such sequences, one or more of which is enabled at the critical radius.**

Also, what about the original problem, Gizmo Gears? Surely that jumbles too, right? Well hold your horses. The image of the same point for R = sqrt(2), N = 12 is actually finite (30 points). My immediate next task is to start stepping and see what happens. I'm sure I'll quickly find a point that generates an infinite set. …"

"Yes, Gizmo Gears jumbles too. Here's an image of theta = 0.1 for N = 12, R = sqrt(2), 25x, 10,000,000 points generated.", (INA) Bob Hearn, page 1

“Hi Bob,
Those images look fantastic! Now we finally know what the proverbial **Bram Cohen jumble dust** looks like. …” Oskar van Deventer, page 1

Page 2 Thursday, August 8, 2013 thru Sunday, Oct 13, 2013

“…I love the way something so apparently simple (two over lapping circles) can produce something with so much order and complexity.” Carl Hoff, page ~2.4

“…N = 2, 3, 4, and 6 don't seem to jumble, but all other N do. Eric noted (on Facebook) that "all values for N that don't produce jumbling are numbers with rotational symmetry that produce tessellations in 2d". Indeed, this is key to understanding what is going on.” “…now I can easily navigate around and change all the parameters without recompiling. Also I can play with color mappings and make lots of pretty pictures. Really, it's way too fun to just sit around and surf through the spaces. If there's interest, I'll share the program (Mac only!)

Again, thanks to VeryWetPaint for first suggesting that jumbling might have something to do with Penrose-tile geometry. It's been clear to me for quite a while that this quasicrystal structure is there, but it wasn't until today that I actually sat down and laid tiles on it. Yep, it works!” Bob Hearn, ~2.41

“Beyond that I can't do more than second your praise (Doug). Amazing, Bob! I am happy that this problem is in good hands.” Andreas Nortmann ~2.42

“…I could see that as one approaches some critical R that the range with a given piece count could get smaller and smaller and there could actually be an infinite number of puzzles with a finite number of pieces”

“I agree with Andreas. I've very happy with the hands this problem has gotten into.” Carl Hoff, page ~2.6

“Thanks to Andreas for doing something with my question, asking him in an email if Gizmo gears had also been checked for jumbling, and thanks to

Oskar for the alert about the Battle Gears effort. Andreas clued me that Battle Gears had been unbandaged…

…My **Circle Puzzler's Manual** talks about puzzles with an **infinite number of pieces,** but did not anticipate the jumbling idea. (CPM did mention Penrose tile circle puzzles and infinite cone circle 3D circle puzzles)" Doug Engel page ~2.7

(The below comments show how cutting edge Bob Hearn's analysis gets.)

"…So -- that's one way to investigate the behavior for N1, N2 without having to worry about which R1 and R2 to look at: set R1 = R2. But there's another way. It turns out that there is a minimum R1 for which, below this value, there is no jumbling even if R2 = ∞. So we can look at this minimum R1, then set R2 to the minimum value for which there is jumbling.

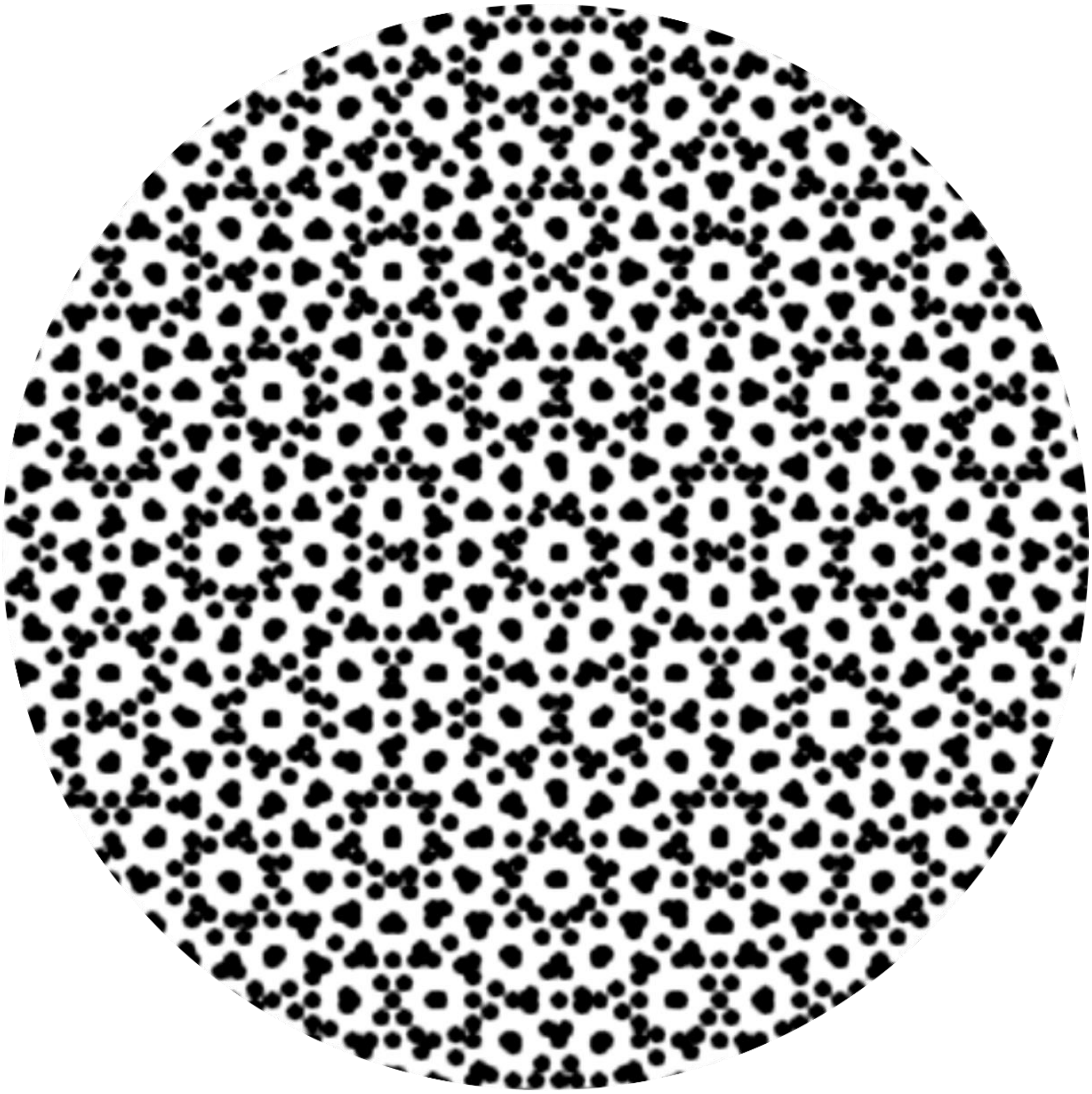
Here is this critical fractal for N1 = 3, N2 = 5, R1 = 2.07 (the minimum jumbling value for any R2), R2 = 3.285 (the minimum jumbling value for R1 = 2.07):

And the full puzzle: (several Bob Hearn images no longer available here)DE

But wait! After playing with this a while, I realized something very cool. Now we can investigate jumbling for two disks, with any N, by finding the critical R. And for two disks with differing N, N1 and N2, we can look at R1 = R2, or R1 = minimum jumbling value.

But what if we wanted to investigate puzzles with more than two disks? I don't want to write that code. The program is complicated enough already, believe me. But guess what... I don't have to. Suppose R2 is large. Then what do we have? We have a puzzle With N2 disks, each with N = N1, R = R1!

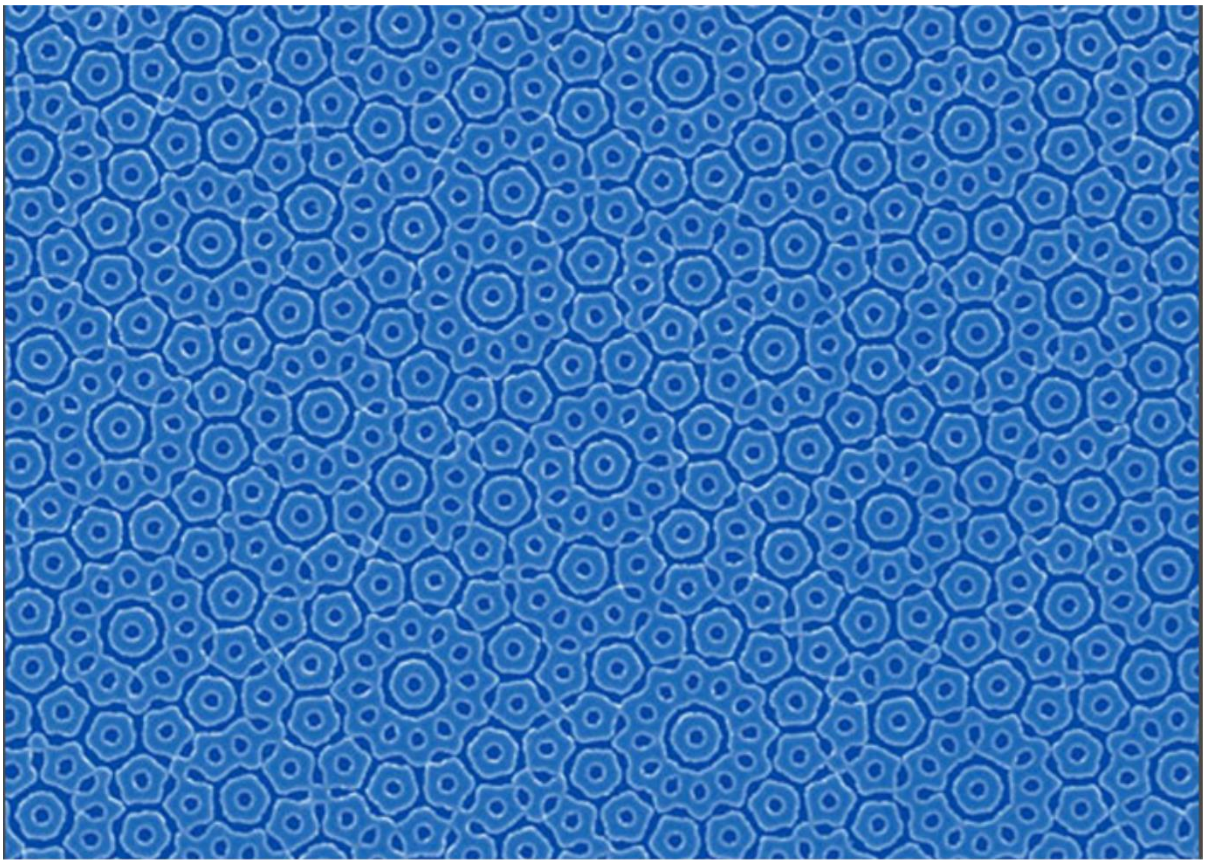
Like this, for N1 = 5, N2 = 4, R1 = 1.805 (minimum jumbling value), R2 = 5 (or anything large). This is a two-disk puzzle, with the stated parameters, but it's exactly the same as a FOUR disk puzzle, with N = 5, R = 1.805, disk centers at (-1, 0), (1, 2), (3, 0), (1, -2):" Bob Hearn, page~2.8 **(If you take the time to sketch this it is easy to understand)DE**



Jason Smith's image of a Moire pattern he made with Mathematica that shows fractal-like behavior and an optical illusion of circles. He pointed out that Moire patterns can show quasicryslike-fractal patterns.

"Thinking of this as a pattern (the cuts) repeatedly overlaying after some rotations, I feel these patterns must be related in some way to Moire patterns." Jason Smith ~2.71

"…Also, Moire patterns display fractal-like similarity at different scales. I need to find an example of that." Jason Smith, ~2.9 (I made this rounded, lots of optical effects.)DE



https://physicsworld.com/a/complex-quasicrystals-created-using-new-nanofabrication-technique/ Physics World, Sep. 6, 2012

(Bob Hearn provided this link to an article in Physics World about quasicrystals after commenting on Jason's images using Moire patterns: https://physicsworld.com/a/complex-quasicrystals-created-using-new-nanofabrication-technique/ (near end of page 2.)

From Physics World 9-6-2012 image and quotes with permission. "Researchers in the US have invented a new nanofabrication technique that can generate 2D patterns with very high rotational symmetries over large areas using industrial photolithography techniques. … Dubbed moiré nanolithography, the technique can produce patterns with rotational symmetries as high as 36-fold – something that has never been observed in nature. … Until the 1980s most researchers thought that long-range order in physical systems was impossible without spatial periodicity. … In 1984 Daniel Shechtman of the Technion-Israel Institute of Technology discovered quasicrystals – materials that have ordered but not periodic structures. … A sample with 10-fold rotational symmetry therefore remains unchanged after being rotated through 360/10 = 36 degrees. … Shechtman bagged the 2011 Nobel Prize in Chemistry for his efforts. …

Now, a team led by Teri Odom at Northwestern University has created 2D quasicrystal nanostructures with a staggering 36-fold rotational symmetry using a new moiré nanolithography technique. "We succeeded in making nanopatterns with rotational symmetries higher than any quasicrystals previously reported by performing two or more exposures through patterned poly(dimethylsiloxane) (PDMS) elastomeric masks," Odom explains. "Because we first make the patterns in a photoresist, we can then transfer the moiré pattern onto a wide range of materials, from silicon to metals. We can then fabricate omnidirectional reflectors or electrodes, for example, using these structures fairly easily."

Another link by Bob Hearn https://www.nature.com/articles/nature12186 Nature May, 2013 (with permission.) Near end of page 2.

Hofstadter's butterfly and the fractal quantum Hall effect in moiré superlattices

- C. R. Dean, L. Wang, P. Maher, C. Forsythe, F. Ghahari, Y. Gao, J. Katoch,
- M. Ishigami, P. Moon, M. Koshino, T. Taniguchi, K. Watanabe, K. L. Shepard,
- J. Hone & P. Kim

"… When subject to both a magnetic field and a periodic electrostatic potential, two-dimensional systems of electrons exhibit a self-similar recursive energy spectrum[1]. Known as Hofstadter's butterfly, this complex spectrum results from an interplay between the characteristic lengths associated with the two quantizing fields[1,2,3,4,5,6,7,8,9,10], and is one of the first quantum fractals discovered in physics. …"

This is interesting because My Atomihedron, a 6 dimensional puzzle, and a precise topological tetrahedron of linked knots, produces a planar projection that looks very much like Hofstadter's Butterfly, when taken to high order, with volumes of millions of linked knots. I once wrote to Hofstadter about it, but he did not have time to look into it then. https://www.puzzleatomic.com/ATOMIC%20pg1.htm

Page 3 Sunday, October 13, 2013 thru Monday July 13, 2013

"…Certainly, wall paper worth and published paper worth!
I strongly believe that somehow this could be related to a more generalized and universal jumbling when we extend the circles to spheres (something which seems very intimidating to me).
And I am convinced that the jumbling we knew for 3D designs would be a small part of this... And Bob, again, wow!!!" Pantazis Kastellorizo, page 3

(This is a … battle between symmetry and asymmetry. Incredibly it shows nice geometric patterns, along with areas that get cut infinitely into Bram's dust. What conditions or what R would cause the dust to cover the entire area of the two circles? This might exhibit some kind of dust pattern where some of the dust is thicker (as infinities go!). These differential dust areas could be delineated by different colors producing more interesting graphics. Bob has already done something like this in some of the first graphics he produced that look like Xray images of crystals.

What defines critical R? Is this a rational fixed number or can it vary? Can it be irrational or transcendental? Does it represent some new mathematical constants for various symmetries of two circles? Is there more than one critical R for a given symmetry of two circles?... Pentazis suggestion of trying this with spheres … …the center of each sphere can turn about an infinite number of axes. … 'then' limit the number of axes (and only two circles) and only allow rotations about a fixed set of axes. … then take a cross section through the two circles…. Read on for answers) Doug Engel

“So... turns out I am giving a talk on jumbling circle puzzles this weekend. I have a lot of material, and only 15-20 minutes to fill, so I should be fine, but I am kind of stressing out. I would like to have some movies, zooming in and/or changing R, which will be very computationally intensive to generate (but I may try), and I would like to have answers to some of the remaining questions. In particular I'm still unsure about the significance of finding quasicrystals and aperiodic tilings here. It may be that the Penrose tiles for N = 5 are a special case... I am going to try to generate diffractograms for some > critical images for other N. That should disclose quasicrystalline structure definitely. But, again I may not have time for this.

Oh, I do have one new thing to mention here. I talked about two ways of exploring critical R values when we are allowed to vary N and R between the disks: (1) keep R1 = R2, then for any given N1, N2, there is again a critical R we can find, and (2) set R1 = minimum critical R for any R2, and R2 at the minimum critical R for that R1. In both cases, we have a clear definition of a jumbling transition, again generating a discrete family of fractals, two (both conditions) for each N1, N2.

But in general, for any given R1 which admits jumbling at all, there will be a minimum R2 which jumbles. So really we have, for any N1, N2, a 2d plot of what the jumbling transition looks like. Sort of like a pressure-temperature phase diagram in physical chemistry.

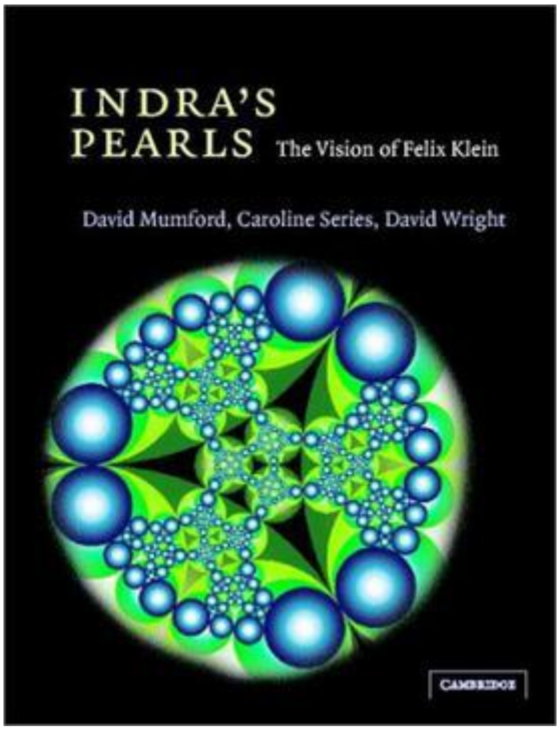
Creating such a diagram here is very tedious -- resolving the jumbling transition can take a lot of time and careful adjusting of search parameters.

But I have started to fill in one such diagram, for N1 = 3, N2 = 5:

R1 is horizontal, R2 is vertical. Each plotted point is a critical (R1, R2) pair, generating a fractal image. Stepping through the images in quick succession, it looks like seeing successive slices in a 3d structure. The interesting parts are the sharp transitions. I don't totally understand them, but it seems to be that different points on the circle hit the fractal transition at different R. But generally, when you are just past a critical R, the generating point quickly starts to fill the space, and smear out, obscuring any fractal generated by another point. Which suggests there are potentially lots of other fractals hiding in here that I won't find by the methods I've used so far. A little more work here should clarify this important point.

The points define a curve: on one side (left/below) we have non-jumbling, discrete behavior; on the curve we have fractals; and above/right of the curve we have quasi-crystalline structure. (graph not posted) DE.

Oh, and one more thing. I was discussing this recently with another computer scientist at a Gardner Celebration of Mind event, and he pointed me to the wonderful book Indra's Pearls, by Mumford, Series, and Wright:



I have been reading this for the past few days. This is kind of exciting, because the situations they analyze are sort of similar, but not the same as what we are doing. If our results here were known, I think they would likely have been mentioned in this book. We explore the orbits generated by turning two intersecting disks; they explore the orbits generated by two Möbius transformations of a particular kind. Actually I had already considered generalizing our problem beyond rotating circles to using general Möbius transformations! However, they use continuous transformations, of the entire plane; we use discontinuous transformations, and it is the points on the circle boundaries that are interesting. We can also imagine using Möbius transformations that "slice" along some curves, the points on one side moving, the points on the other not moving; that is what I had been wondering about. What we have now is in fact a special case of this more general problem.", Bob Hearn, page 3

“Any details on the forum? Is this something open to the general public? Would this be something you could record and upload to YouTube for example? I'd LOVE to be able to see it. The movies sound great too but I'd love to see them in the context of the presentation.” Carl Hoff, page 3

“It's the Hacker's Conference. Unfortunately no, the conference is invitation only, and not only is it not open to the public, but everything that happens there is off the record; no recordings allowed. Alas. I gave the "Fireside Chat" (equivalent of keynote) there a few years ago. I really wish I had a recording of that.

It's not really like it sounds; there's nothing sinister going on. This is about hacking in the original, creative sense, not the computer criminal sense. The conference decided to take a lower profile long ago when CBS news did a segment on them. They were led to believe it would be a positive piece, but it was a hatchet job. So... now it's all "secret". Shhh.

The good news is, this will be the first talk of at least a few; hopefully I can polish it as I go. Probably I'll give a more condensed version. And I'm giving a CS colloquium at a local college next spring as well, where I may talk about this, depending on where the project sits then. …”, Bob Hearn, page 3

“To add to Brandon's suggestions. I'd be curious to see some estimate of the size of the smallest pieces. Assuming the base circles are say 10cm in diameter at what point do the smallest pieces become of order the cross-sectional area of an atom?
I'm also curious what the transition at the critical radius looks like in an animation like this? Is it obvious once its reached?
I'm really looking forward to seeing the talk at G4G11.” Carl Hoff, page 3

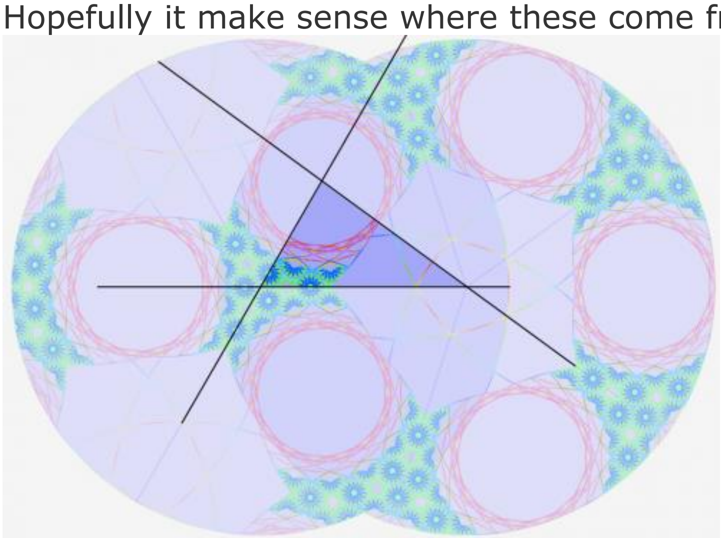
“I'll try to get to your questions soon, swamped right now. But here is another one, a zoom of the N = 7 fractal:
http://www.youtube.com/watch?v=rv2lKlqo_wQ
(zooms in to show fractal self similarity)
This is what happens to the previous movie right at the critical radius (1.623579), at least the part of it that goes infinite.”, Bob Hearn, page 3

“Also, any chance you can determine the critical radius to much more precision? It's a long-shot but I'd like to run it through one of those algebraic number brute forcers to see if it's something like sqrt(2)*sin(1/3) or something simple-ish like that.”, Brandon Enright, page 1

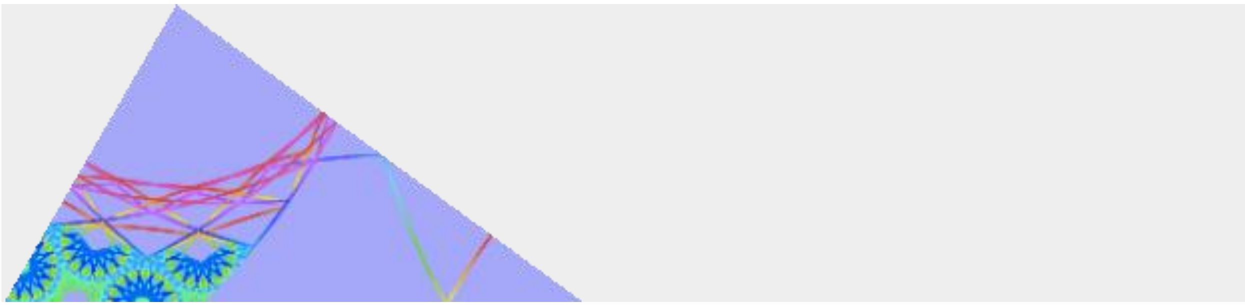
Twisty Renderer -- Jaap's Spheres with POV-Ray

“What could be happening is that it may be possible to make a finite puzzle with any arbitrarily large (but finite) number of pieces as one gets arbitrarily close to the critical radius. This would mean there are an infinite amount of finite sized puzzles and the critical cuts are likely buried an infinite number of iterations deep. …” Carl Hoff, page 3

“…I was doing the same thing with my program. Looking at some of your more recent pictures, there's actually a few more lines of symmetry we might be able to use. I've drawn them in over your n1=3, n2=5 example. Hopefully it make sense where these come from



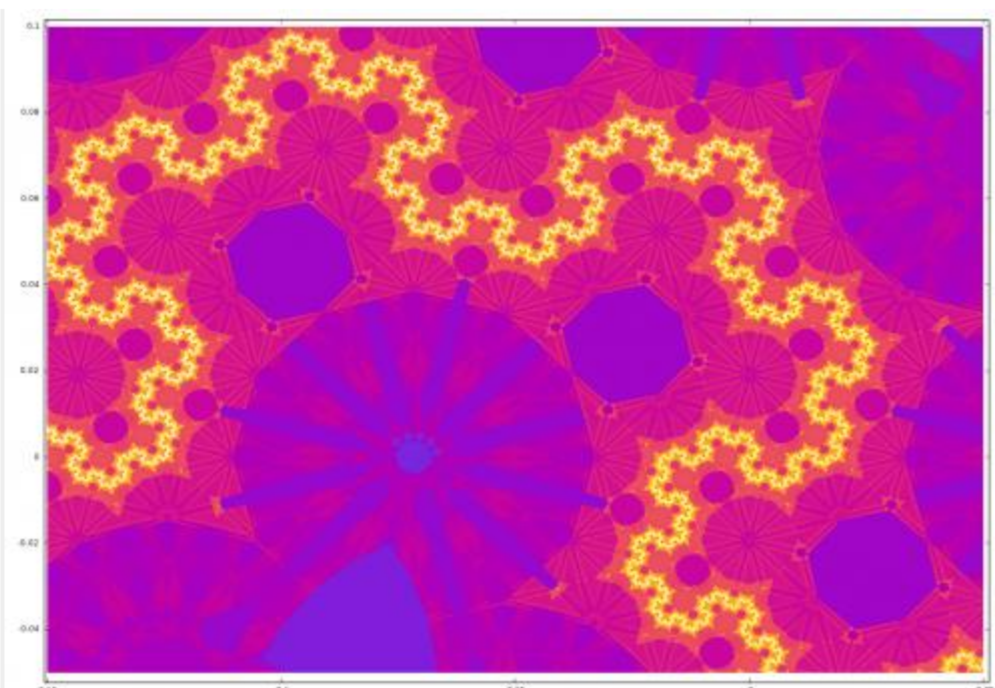
This leaves just a triangle as the critical area. Obviously it loses its left/right symmetry because n1 NE n2.



35triangle.png (24.46 KiB) Viewed 955 times

It's cool seeing the arc on the right side bounce around. Makes me wonder if you can do all these computations using just mirror symmetry. …. … I think this shape may help explain the Penrose tiling. For the n=5 puzzle, the triangle has angles 36-36-108 which is half of the dart Penrose tile. And if you're mirroring and copying this triangle around, it seems almost natural that it would create Penrose tiling patterns. …" Guilty Bystander, page 3

"…. So I decided to render a zoomed in portion of the fractal curve so you can see how it's self-similar:



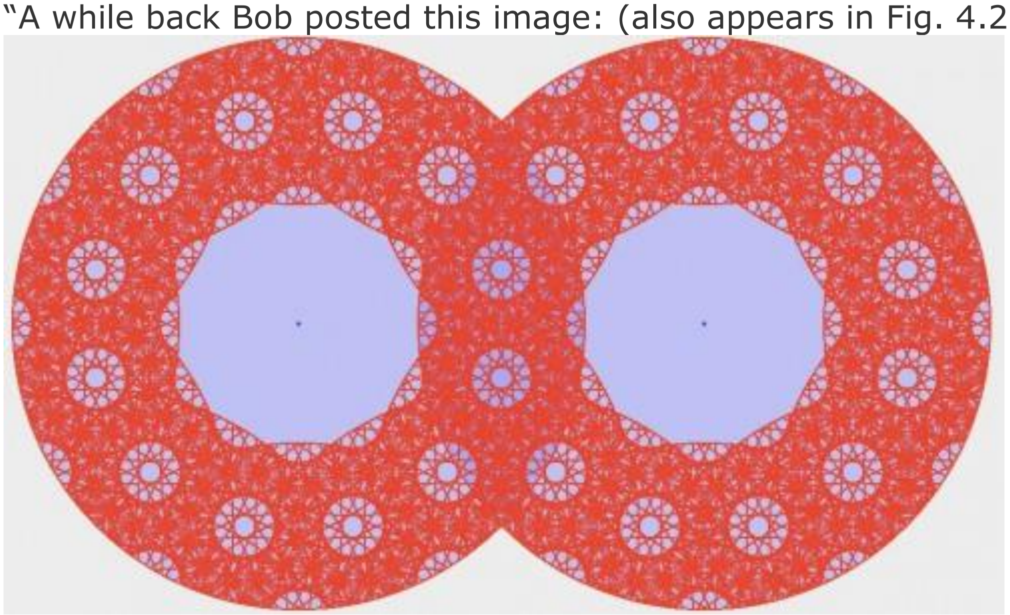
So here is what I think I know:

- The Gizmo Gears Jumble (though Bob already convinced us of that)

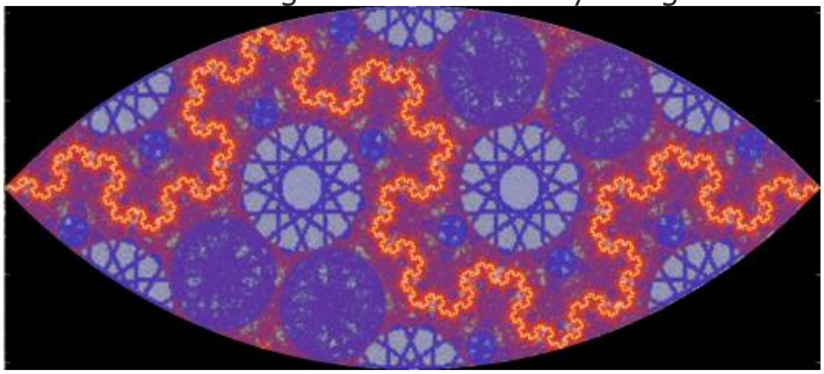
- [A B]xN is not sufficient to trigger jumbling
- [A' B]xN does cause jumbling
- There is a boundary curve and the closer a point is to the curve the higher its order is
- Points on the boundary curve (I think [0, 0] is on the curve) have infinite order
- The boundary curve is a fractal similar to a **Koch Snowflake**
- The curve isn't mirror symmetrical about the Y axis because A' B isn't mirror symmetrical about the Y axis …. ", Brandon Enright, page 3

"… but I guess that in some respects it is obvious. Aren't [11A B] and [A' B] the exact same thing? And it too looks like Bob struggled to find a single generator sequence which would give him an infinite orbit. Looks like you found one. I need to go back a re-read all of Bob's posts as I'm not certain he ever did at the moment. …" Carl Hoff page 3

"A while back Bob posted this image: (also appears in Fig. 4.2)



I cut out the wedge and overlaid my image showing how they match:



Note that mine only goes through half because I'm only looking at [A' B]xN and not [A B']xN too. If I combined both my image would have the mirror symmetry you'd expect.", Brandon Enright, page 3

"Wow!!! Very cool stuff, Brandon. I am very surprised, also embarrassed here... indeed, I had convinced myself no simple generator sequence on its own would cause jumbling. I don't know how I missed this.
"… but how can everyone have missed this simple way to generate the snowflake? We will have to check carefully to make sure that indeed this wasn't known. The snowflake has been very well studied. Perhaps this hasn't been discovered because doing it your way, which is similar to the way fractals like the Mandelbrot set are generated, would seem to require the very high-precision numerics you have used.

Brandon Enright wrote: "I have a hunch I know what determines the critical radius. I want to explore some variations in parameters to test my hypothesis. If I'm right, I'll do my darnedest to show what's happening."

A few months ago I did show my work to Veit Elser, one of the experts I was mentioning in quasicrystals and fractals. It looked novel and interesting to him, and he suggested I publish in **Discrete & Computational Geometry**. I still have not made time to start writing the paper (also I was still hoping to understand the jumbling transition better). Brandon, we may want to talk about collaborating. …", Bob Hearn, page 3

" …WOW!!!!! That sure is a big number 1.97×10^40. To put that in perspective the number of states of a standard Rubik's Cube is 4.3×10^19. So using your calculation **here**. If the diameter of the circles is the same

length as a face diagonal of a Rubik’s cube, and we can make turns with no delay between them and we can instantly accelerate the circles such that their very edge is traveling near the speed of light. We can ask ourselves how long will it take to restore the solved state just making [A B] turns. The answer is greater than 289.4 years times the number of permutations of a Rubik's cube. (it would take) 289.4×4.3×10^19 years
That just boggles the mind. We could have set our speed of light toy in motion back at the big bang and it still wouldn't have made a dent in this problem. …” Carl Hoff, page 3

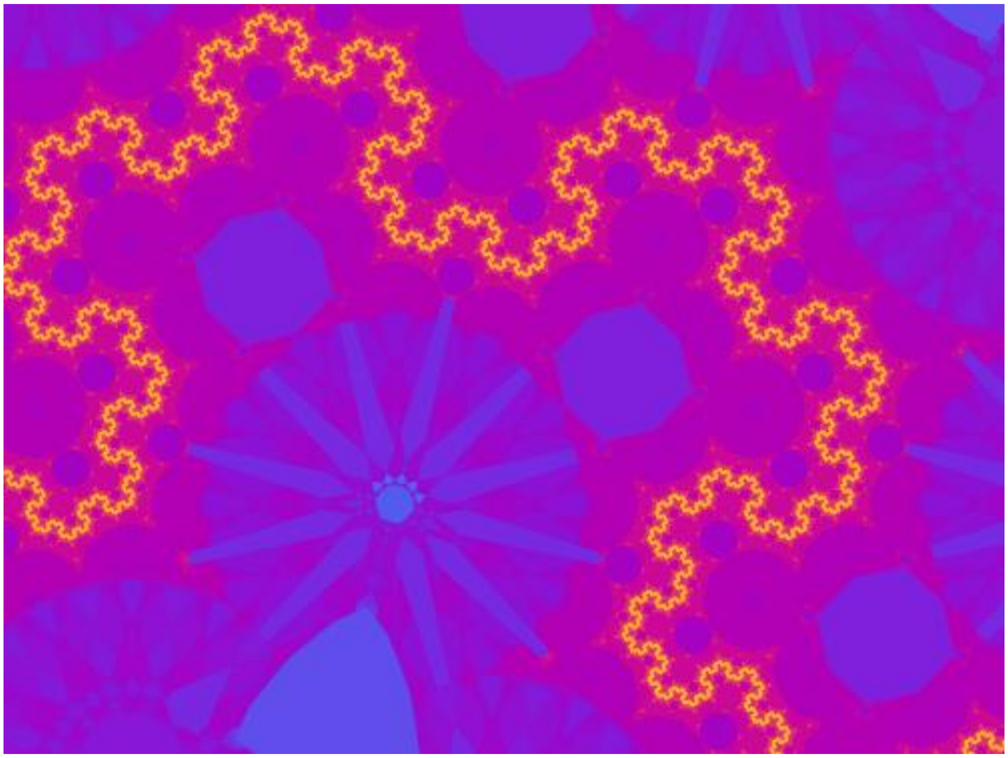
Page 4 Monday, July 13,2015 thru Sunday, July 19, 2015

“Alright Carl, I spent all night working on automating a lot of the image generation process with the goal of one day being able to generate a wallpaper or animation.

The image generation code now:

- Doesn't use gnuplot
- Detects the minimum and maximum orders and scales the color range accordingly
- Locates pixels without samples and generates a new sampling script to fill in the gaps
- Does edge detection and finds pixels that need multi-sample anti-aliasing and generates a new sampling script to with stochastic points for the MSAA algorithm

I made the "zoom" image larger and let my script do the sample selection to fill in the gaps and anti-alias edges. Here is the result:”, Brandon Enright, page 4



“Brandon Enright wrote: “I played with N=7 at the critical radius r=1.623579 today.
At this radius, [A' B] isn't sufficient to trigger the jumbling. The smallest order for points is 28 and the largest order is 40656.”

Post Posted: Thu Jul 16, 2015 9:00 am “Verrrrry interesting! And also a bit of a relief. So I'm not crazy for not finding that transition myself when I was looking for it.” BH

“I find it a bit troubling that there seems to be the start of a fractal pattern yet it isn't jumbling. There was the same "issue" below, at, and above the critical radius for N=12. There wasn't a big change in the pattern but there definitely was an explosion of the order of some points above the critical radius.

It makes me wonder if the fractal pattern isn't actually proof of jumbling. Suppose these puzzles don't actually jumble but there are certain boundaries that get crossed when you increase r that cause an exponential (but finite) growth in pieces and it just happens that the growth follows a fractal pattern. Then the critical radius estimation is not the point at which jumbling starts but rather the point at which there is an explosion of pieces greater than what we can hold in computer memory or time we can wait for the finite order to cycle back around." BE

"Well. It's true that the apparent fractal pattern is not literally proof of jumbling. Based on everything I have seen in how the patterns behave, though, there is a very sharp transition at the critical radius. Below that the fractal pattern develops more and more fully the closer you get. Past that, it starts to smear out and look qualitatively different, as shown above for N=12. I am pretty convinced. Of course it would be much better to have an explicit proof. I was hoping the generator pattern might help with that. It still might... it's starting to look likely that at every critical radius, there is some particular (generally simple) generator that starts to jumble. That gives us something new to look for, for every N. I wonder if there is any kind of pattern?" BH

"I found that [4A, B]xN does seem to jumble, or at least some of the points have a very high order." BE

"Curious. I don't think that's something I've tried. I need to re-activate my generator code (soon!) and see what I can find here as well. Again, I'm optimistic we can resolve the jumbling transitions to much higher precision then, and perhaps search for algebraic expressions for them." BH

"I did a preliminary test.
With N=7 R=1.623579 and [6A B]xN the maximum order was 40656. I changed R to sqrt(2.63600876924) which is 1.62357899999969203839172[...] and irrational and *extremely* close to the original R. The new maximum order was 41545.
I find it interesting that decreasing the radius by such a tiny amount

increased the max order. It still didn't cause jumbling though. I think I'd call this test inconclusive.” Brandon Enright, page 4

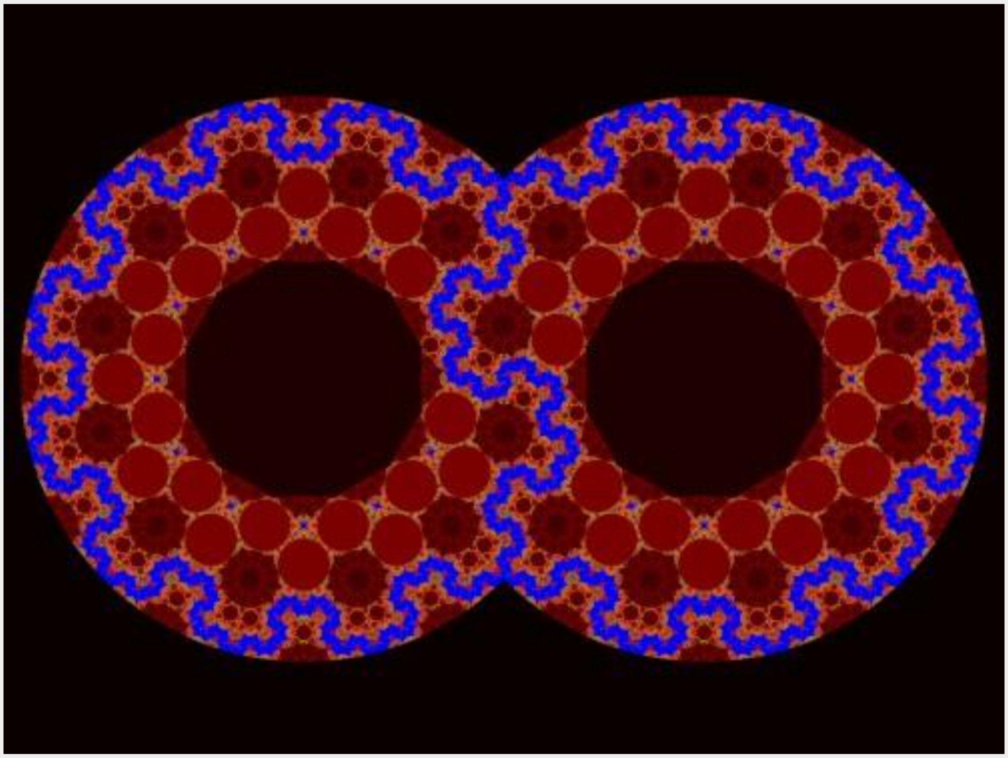
Comment by Doug Engel: This makes perfect sense. As the radius increases at certain places if three or more arcs all meet at point or cross at a single node point the piece count drops since those arcs no longer make separate cuts when they come together at a single point. This means that if the number of pieces is large the piece count can drop dramatically. From this there could be a possibility that jumbling could stop at single R sizes when getting to such an arc meeting spot (considering symmetry such crossings will happen at several places simultaneously). This is discussed above in Fig. 5.1.

I believe it to be a supremely important consideration with these fractals and may have a lot to do with them suddenly making huge changes in appearances. Also discussed in Fig. 5.1 when two arcs become tangent the piece count increases, by 1 per tangency, and then as the same arcs get larger and cross over each other the piece count increases again by 1 per cross over.

Therefore I suggested at some point in an email or in the discussions to write a program that shows all the completed circles (or circle centers) outside the two-circle puzzle as the cuts and moves proceed. It would show the circles moving along with pieces they contain. The centers of all these circles would form another fractal with a very interesting geometry.

“With Brandon and Bob's kind permission, I wrote a toy version of what they've been doing. Here's a full render of the Enright Snowflake.

Very interesting that it isn't "just" a Koch Snowflake, isn't it?” Jason Smith, page 4



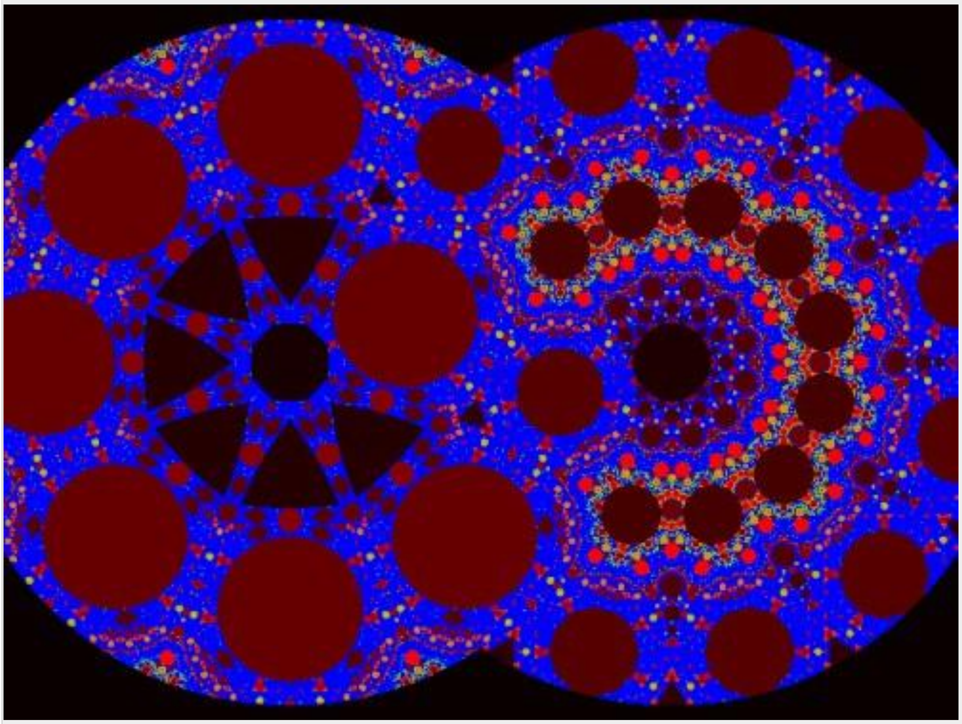
“That looks really cool! Your code must be several orders of magnitude faster than mine. No doubt Oskar already has three different gearing mechanisms in the works using the snowflake curve!” Brandon Enright, page 4

“Man, so much cool structure that I missed, all because I convinced myself early on that no simple generators jumbled. I should have revisited that assumption. But now I'm excited to see what else turns up!” Bob Hearn, page 4

“… In case anyone is interested, the version I'm working with is written as a shader in my ray tracer. I'm not doing anything clever beyond wrestling C++ into compiling, unfortunately. So the speed must be a combination of compiled C++ and Brandon's simple A'B iterator. My ray tracer just made it a bit faster to develop since I already had ray generation code, and image and color libraries, etc.

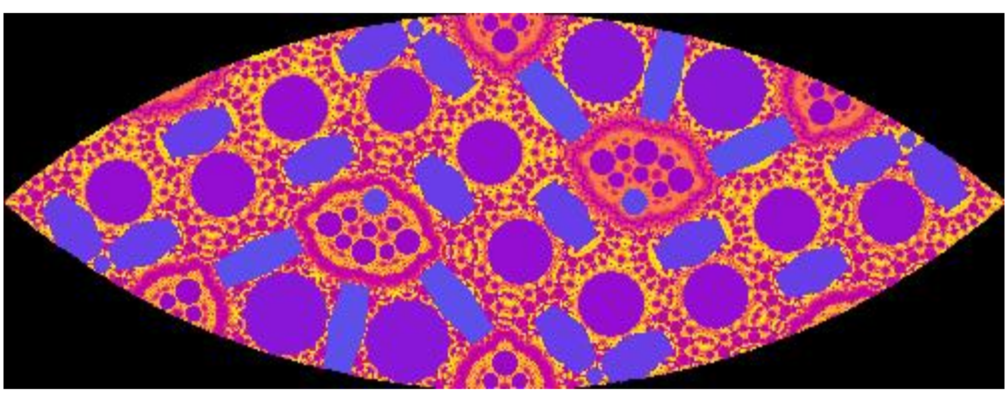
My ray tracer takes text files as input, so I may be able to give out an exe at some point for people to play with.
The code supports arbitrary numbers of discs with their own properties.
Here's one with N8 , N12, r1.8, max 5000." Jason Smith, page 4

Attachments:



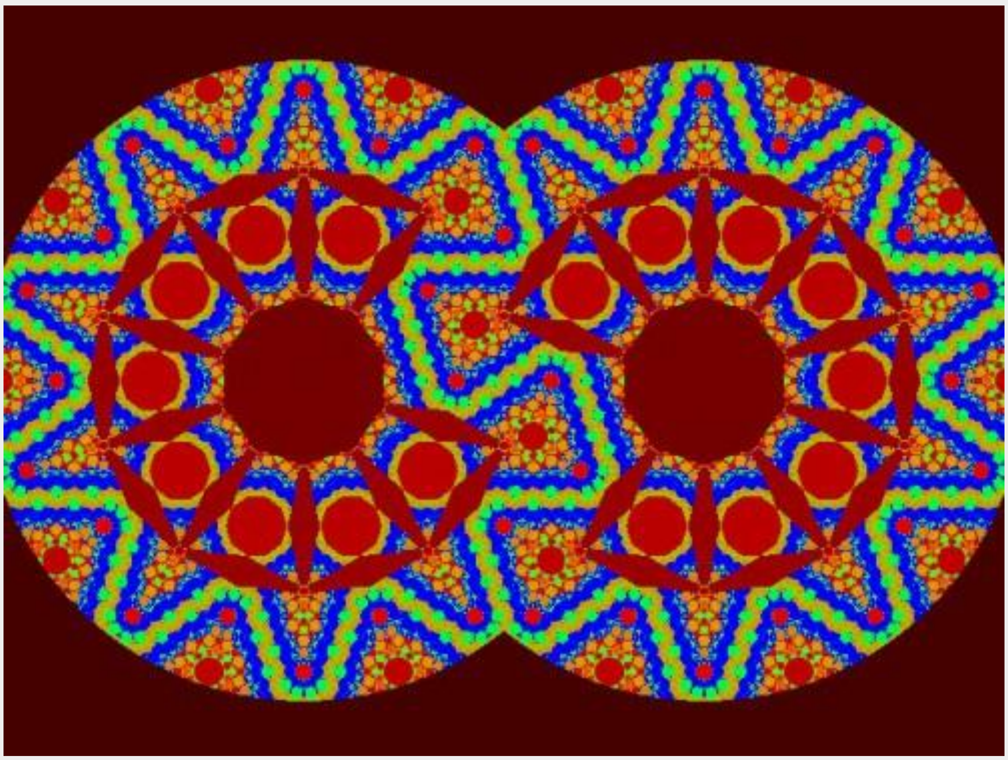
Page 5 Sunday, July 19, 2015 thru Tuesday, July 21, 2015

"I tried N=14 with G=[13A B] at R=4/3 and it jumbles very deeply very fast.

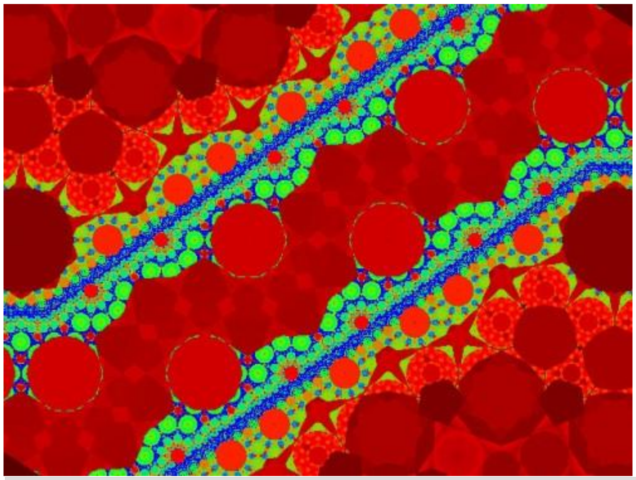


A significant amount of the area in the wedge has orders > 100 M so I had to keep trying to sample points over and over until I found a point for each pixel with a smaller order." Brandon Enright, page 5

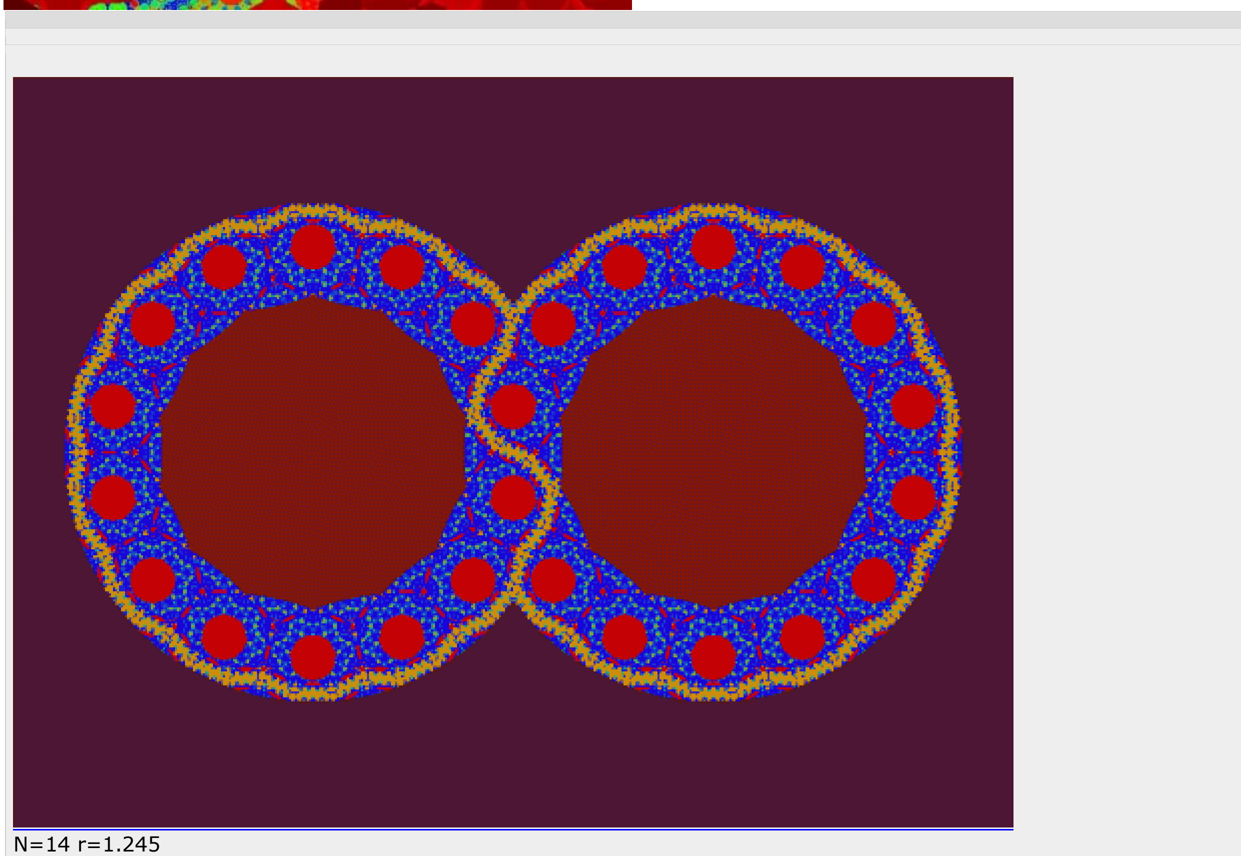
"That's very cool. Here's N=10 at r=1.6 I'm going to try to get closer to the threshold." Jason Smith, page 5



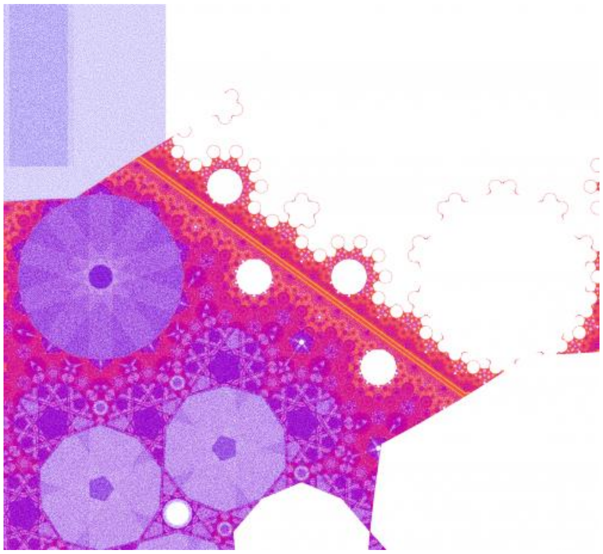
"Oooh so cool! Something happens between 1.5 and 1.6 to make the "gear tooth" curve. If you can narrow in a bit on the radius where that starts to form I'll render it with my code." Brandon Enright, page 5



“Yeah, Brandon if you can see one of these to run through your code I think that would be a good idea. I decided to see if anything snapped into form when the blue lines converged. So zooming in and sampling in the range 1.54 to 1.6. Pretty but nothing too surprising.” Jason Smith, page 5

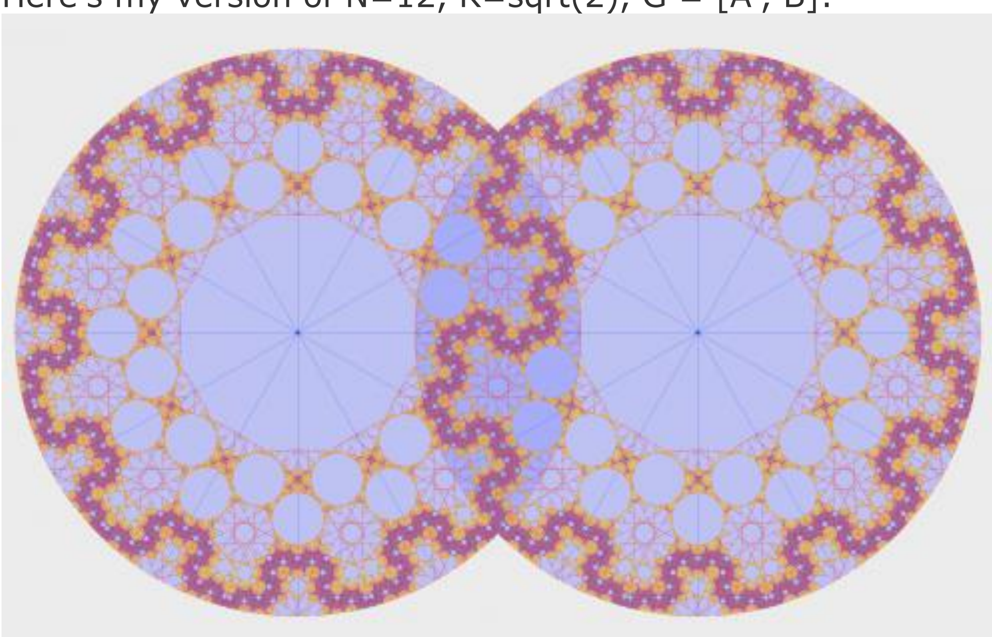


“I'm about to call it a night. Here is the progress so far on zooming into N=10, R=1.545, G=[A' B]:” Brandon Enright, page 5



“Great work, guys. I'm trying to reconnect my generator code, but the rest of my code now has too many assumptions in it that are violated by what's needed to track a generator. Mostly, segments are not allowed to leave the intersection. Plus it's got lots of, well, stuff, lots of options, and I've forgotten how it all works. It's been a couple years. Meanwhile Jason has created a great program from scratch!” Bob Hearn, page 5

Here's my version of N=12, R=sqrt(2), G = [A', B]:



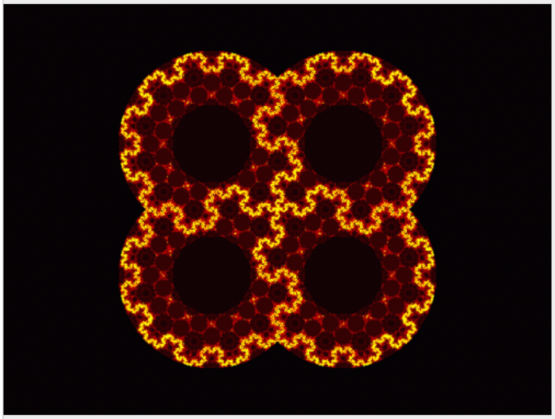
“… This time, I have colored by depth. This matches what you guys have, but is not as pretty, because the piece interiors are not colored; OTOH, the piece boundaries are sharply defined, because I'm just imaging the disk perimeter. There are definite tradeoffs between our two approaches, when rendering generator patterns. I only have to image the perimeter, and you have to image everything; but if I want to zoom in, I have to image the perimeter to very high precision, and store all the image segments, whereas you still only have to image the number of points you actually want to display. …"

"… I am realizing now that my original generator code never worked: it assumed that you can ignore segments that move outside the intersection. **This is true when considering all moves, but not when following specific generators.** As a result, I missed all this wonderful structure! I can't stop surfing through all the spaces. When looking at the full set, once you go past the critical radius it all gets pretty smeared out. But when restricted to a generator, you just get more and more interesting patterns. …" Bob Hearn, page 5

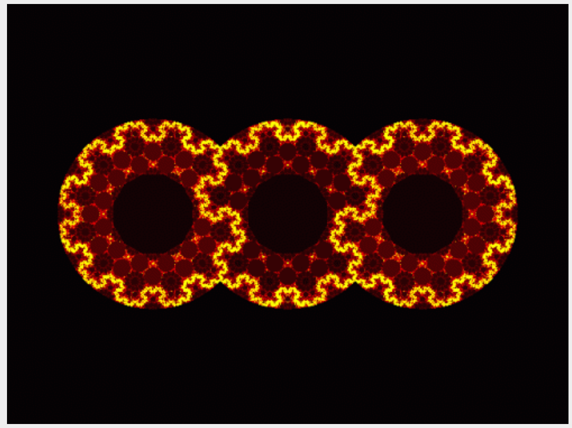
(This makes sense. Think of how a Rubik cube can generate symmetrical patterns when repeating a simple generator or following an algorithm. The problem and the joy is that we now have infinity times infinity of tile like fractals to choose from, how do you choose the best one!! It is mathematical art unleashed) DE

Brandon Enright wrote: "Here is the progress so far on zooming into N=10, R=1.545, G=[A' B]:"

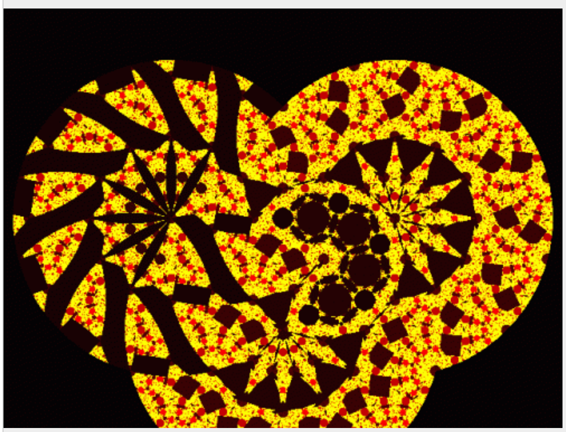
"After a bit of housecleaning in my code, I thought I'd try multiple discs. Attached are experiments with 3 discs and 4 discs. The last two are interesting because the rotation of the third disc determines whether everything falls into jumble dust." Jason Smith, page 5



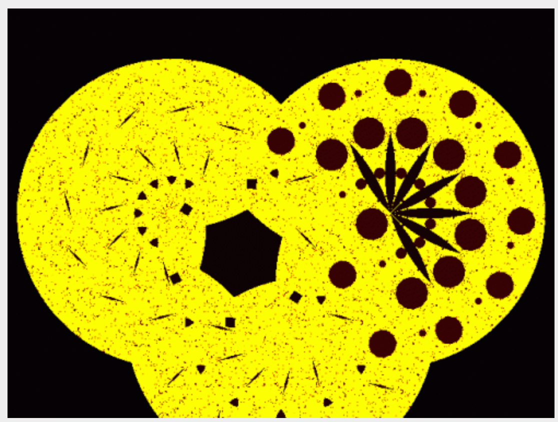
N12 4 discs= grid [A' B C' D]



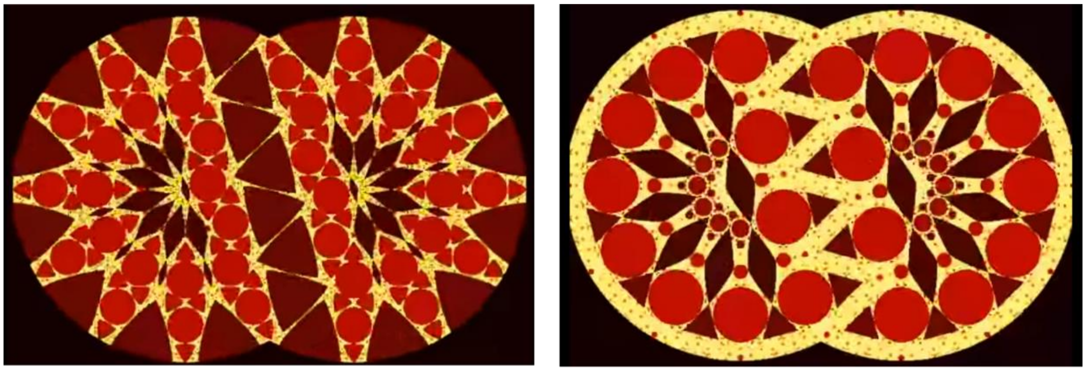
N12 3 discs [A' B C']



N12 Cwise third



N12 CCwise third



Gizmo Gears Fractal for N12, Jason Smith, page 5
https://www.youtube.com/watch?v=yGPtgl9r4Ew&feature=youtu.be

Page 6 Wednesday, July 22, 2016 thru Thursday July 23, 2015

http://www.youtube.com/watch?v=5Nx7xd1tC34 Video for N10 R 1 to 3 Jason Smith, page 6

"… Imagine though as point 4 moves through the wedge. It first moves into the wedge after the move A' is done and then it gets moved up some when B is done…. …Now if the distance between 1 and N isn't a rational factor of the length of the line segment (and with some more assumptions) we can show this process repeats infinitely.

… Now that we're dealing with lines (Bob Hearn fractal) and they have definite lengths it seems more tractable.

So some places I think we could start:

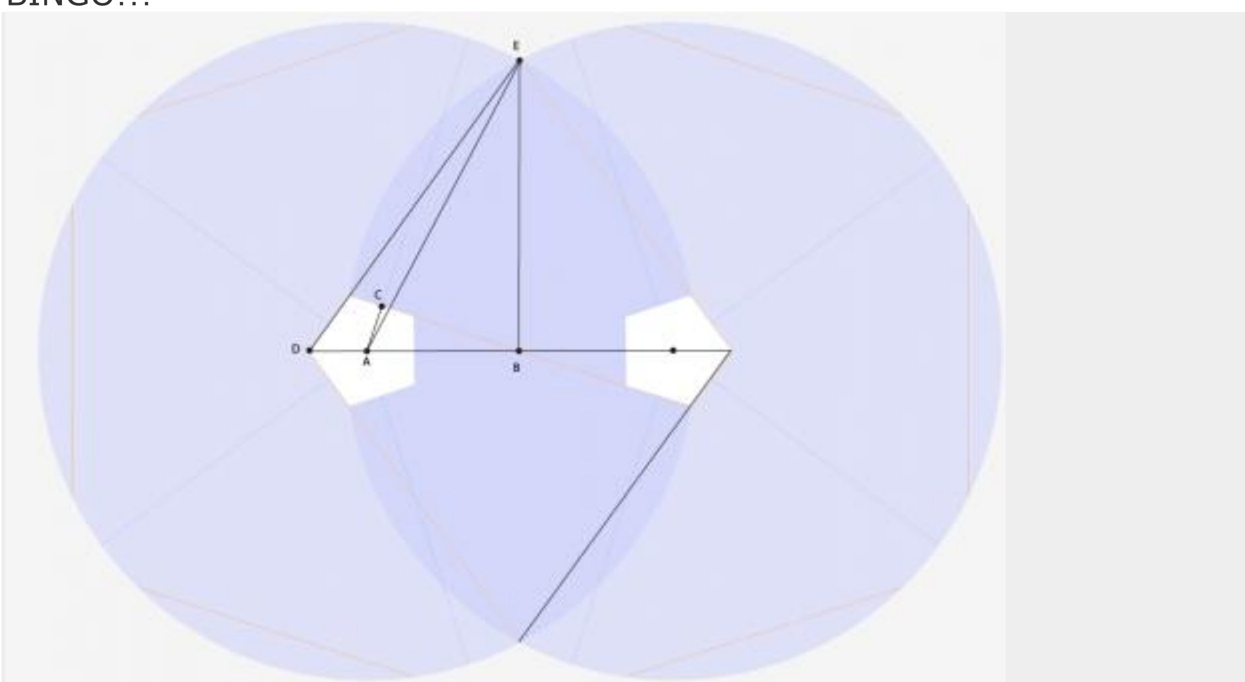
- Figure out what condition must hold for the zig-zag line to form in the first place (this would tell us the critical R)
- Figure out what happens to point 4 before it ends up landing back on the first line segment as point N
- Show that when the point is moved through the wedge by some number of B moves the point "skips ahead" or "falls behind" by an irrational multiple of the line segment length

…Maybe we can setup some equality or set of conditions involving these points that only holds true for one R and then solve for R?" Brandon Enright page 6

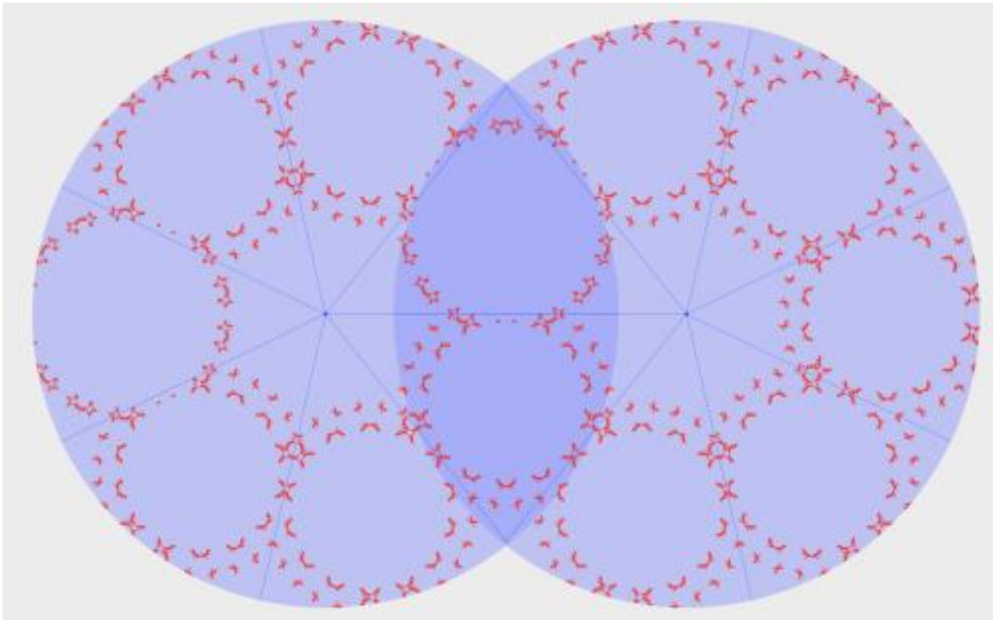
"OK... I've got it!!! The first closed-form expression for a jumbling radius! With reference to the following figure, which is the critical jumbling pattern for N=5:
The pattern of lines exactly fits the displayed pentagons, from which we can deduce R!

AB = 1
angle ABC = pi/10
AC = sin(pi/10)
AC / AD = (1 + sqrt(5)) / 4 (pentagon geometry)
AD = sin(pi/10) / ((1 + sqrt(5)) / 4)
BD = 1 + sin(pi/10) / ((1 + sqrt(5)) / 4)
angle BDE = 3 pi / 10
BD = DE cos(3 pi / 10)
DE = (1 + sin(pi/10) / ((1 + sqrt(5)) / 4))) / cos(3 pi / 10)
BE = DE sin(3 pi / 10)
AE = R = sqrt(1 + BE^2) = 2.14896114175 Bob Hearn, page 6

BINGO!!!



"…Here is the first infinite pattern created at N=7 for [4A,B]:

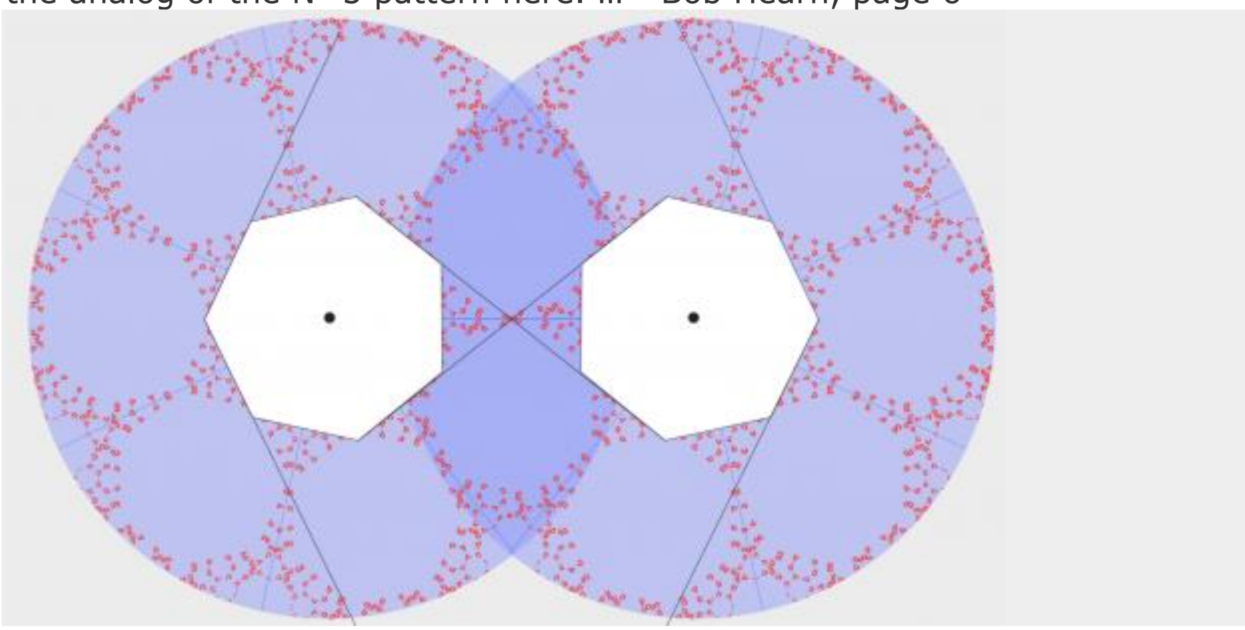


This looks like a patchy subset of the full fractal generated at the critical radius. (Actually, this is just the image from a point on one disk; adding a corresponding point on the other might add something.) Very different from N=5! I kind of think this will be the more general case. I think I have already looked at N=12, and it was also messy. We need to find the... whatever we call them... critical jumbling generators and corresponding point images, for all N. It does appear that N=10 is (mostly) made of straight lines, though not quite in the mold of N=5. Maybe something special about multiples of 5?
I think this is a task that, at present, my program is best suited for, but have you guys thought of adding a threshold, so you just render points with very high order? Then you could also look for the generators and patterns, starting with my table of critical R for lots of Ns -- here's the link again. (BTW I list the critical radius for N=10 as between 1.543 and 1.544, but I could be off slightly.)" Bob Hearn, page 6
https://docs.google.com/spreadsheets/d/ ... iylhao9QfI

"…I don't think that proves jumbling though even though I think we're all basically certain that this set of points is infinite. To prove the jumbling I think we still have to show that the movement in the wedge shifts a point by an irrational fraction of the line length. …" Brandon Enright, page 6

"…I agree that it doesn't prove jumbling, but we now know basically what the process is, and why the radius is what it is. And I think it's now more fair than ever before to say that we have clearly *observed* jumbling, at its most fundamental level. What remains is to formally analyze the dynamics of the system -- we need to show is that this system is **ergodic**. …" Bob Hearn, page 6

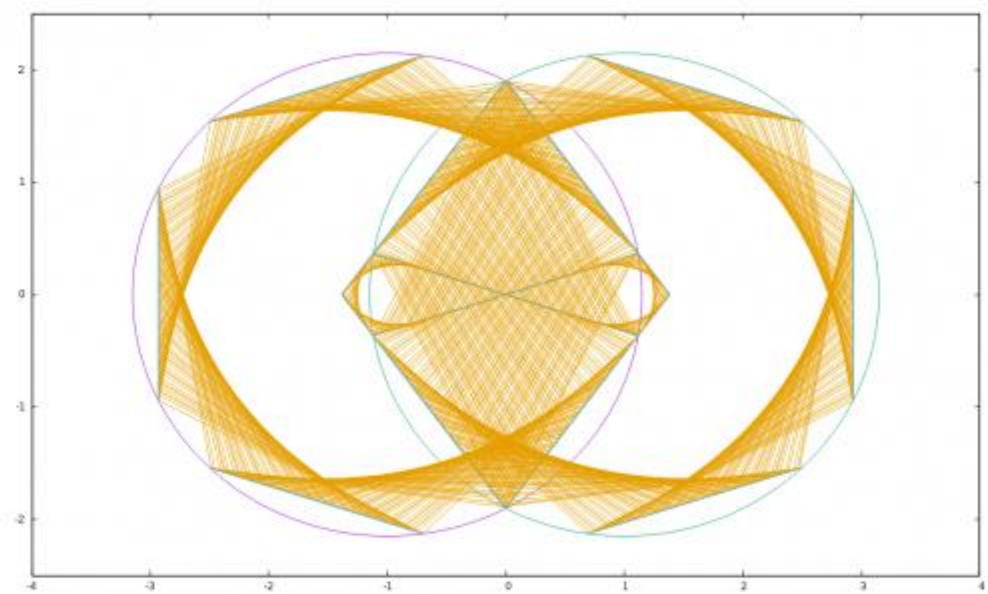
"…If I try to impose the same construction on the figure above I get this, which doesn't work at this radius. Actually, I'm not really sure how to draw the analog of the N=5 pattern here. …" Bob Hearn, page 6



"…Using my system -- imaging the disk perimeter, saving all the segment images -- it's a matter of playing with the radius, the minimum theta step size, and the maximum number of cuts (segments) to store. If the image of any one segment goes over this, then it's "jumbling" at the resolution I'm looking. To get high precision you have to use very small step sizes and very high MAXCUTS. But this means lots of time and lots of memory. In practice I will gradually home in on where the transition appears to be, including finding a particular theta neighborhood that generates a jumbling image. Then when I need to tweak the radius, I can search in just a small theta range, with a small step size. …" Bob Hearn, page 6

"So, here is an animation of the first 200 steps the origin takes for N=5 at the critical radius:

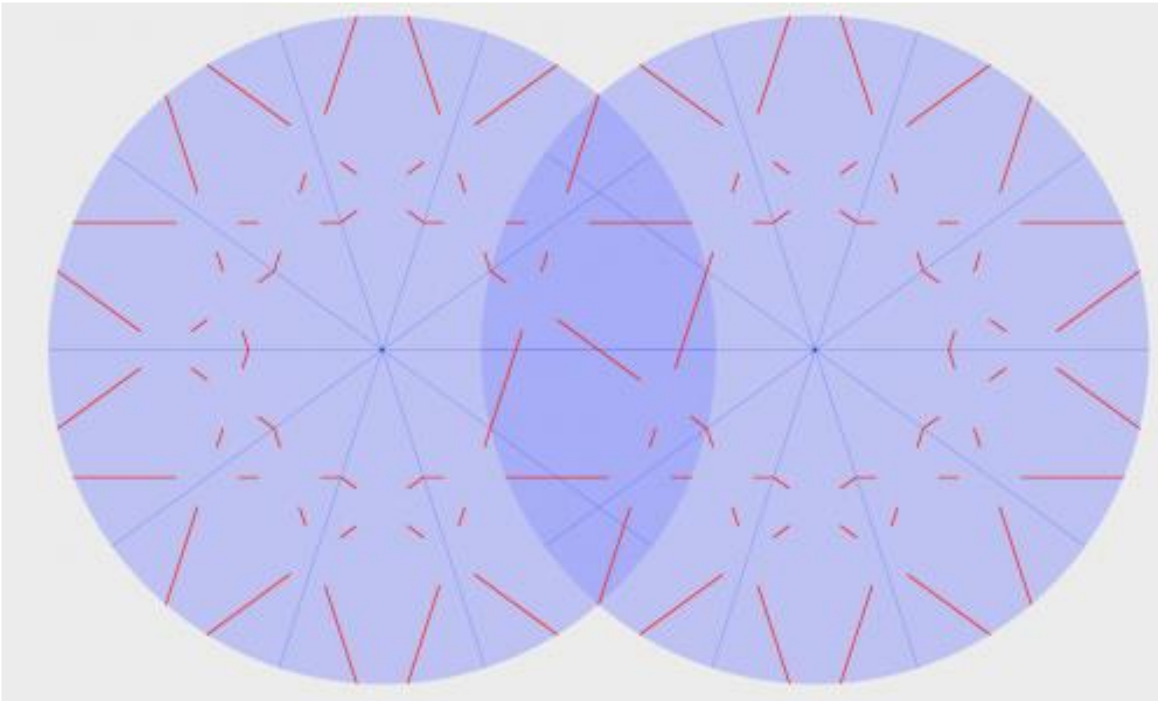
The space gets covered nice and uniformly (after 800 steps):



It doesn't seem to be trivial to prove this is an infinite process however I think we can enumerate all the rules and take it step-by-step:

- A point near the center of a line segment that doesn't fall into the small "loop-arounds" that stick out of the wedge will alternate between orbiting the left and right circles
- A point that orbits the left circle that then falls into the right loop-around will orbit the left circle again without switching circles
- The same is true for the right circle and the left loop-around

I think we could first look at a point that alternates circling left, then right, then back to left without hitting the loop-arounds to show there is some irrational shift. This might need to be done "modulo" the line length so that the shift wraps around when it goes off the end. …" Bob Hearn, page 6



“OK, looking at N=10 at R=1.5433, there is similar geometric reasoning to the above for N=5, leading here to R = sqrt(14 - 2 sqrt(5)) / 2 = 1.54336191843. I'll go through the details if anyone would like them. So, that's two critical radii we know exactly now.
Also, going back over my math from last night for N=5, it can be simplified with various trig identities to R = sqrt((7 + sqrt(5)) / 2) = 2.14896114175. So... yeah. If we had had enough digits, we might have been able to find those expressions with algebraic solvers. So now the expressions are not merely closed form, they are algebraic.” Bob Hearn, page 6

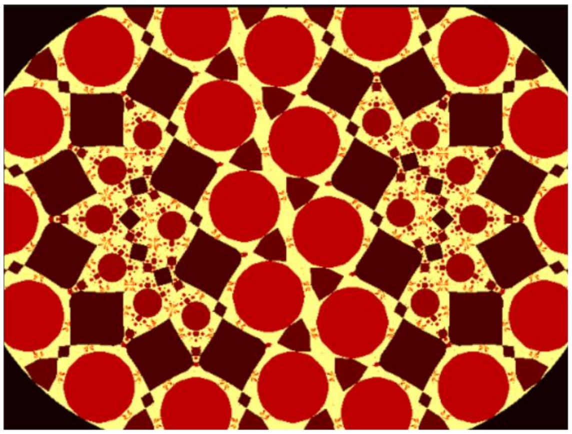
“This was about what I expected it to be. A crazy experiment transitioning from N10 to N12 at sqrt(2) radius. Non integer N's could be interesting to look at in the future, like 6.5 for example.” Jason Smith, page 6 YouTube videos showing how different [A’,B] is from [A,B]
http://twistypuzzles.com/forum/download/file.php?id=49113&mode=view

http://twistypuzzles.com/forum/download/file.php?id=49112&mode=view

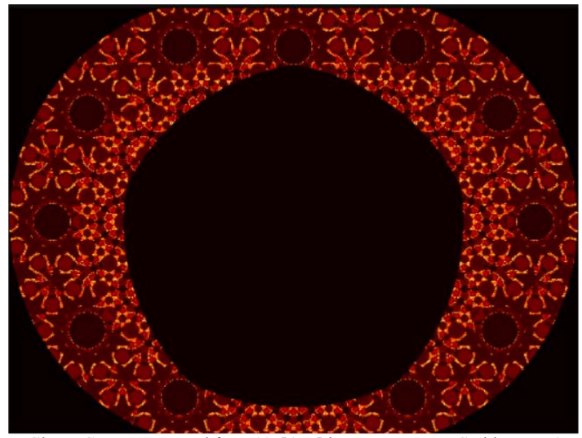
http://twistypuzzles.com/forum/download/file.php?id=49111&mode=view

http://twistypuzzles.com/forum/download/file.php?id=49110&mode=view

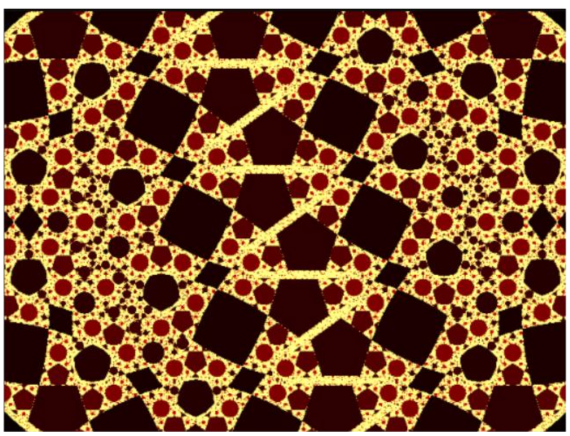
“Haha! Wow. That never occurred to me before. Of course we'll always get jumbling when N is irrational.” BH, p6



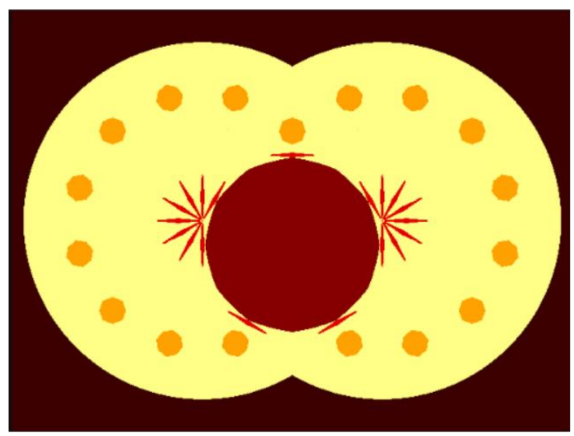
Gizmo Gears UT Fractal for N10, [A’,B] last Image, Jason Smith, page 6



Gizmo Gears UT Fractal for N10, [A,B] last Image, Jason Smith, page 6

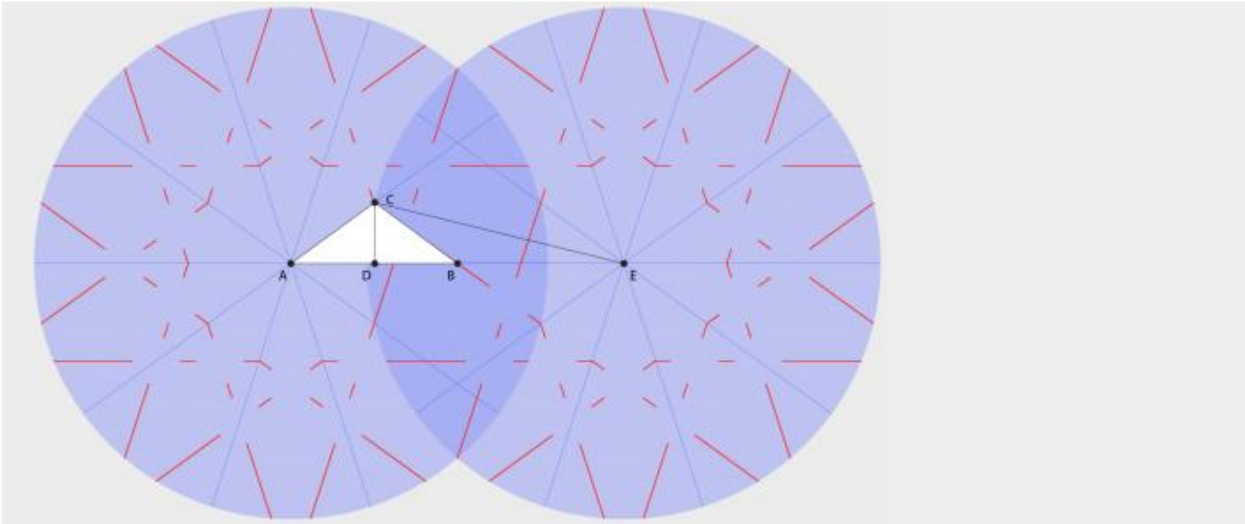


Gizmo Gears UT Fractal for N12, [A’,B] last Image, Jason Smith, page 6



Gizmo Gears UT Fractal for N12, [A,B] last Image, Jason Smith, page 6

“This one is actually simpler than N=5. The key observation is that triangle ABC is isosceles. So AD = DB = 1/2, angle CAB = pi/5, from that we can get CD, and from that CE = R. Again, this line pattern only works at this radius. The zigzag through the middle wouldn't match otherwise. …” Bob Hearn, Page 6



“I'm thinking there may be multiple "critical radiuses", except I don't know if they are infinite. I don't suspect it really. But suffice it to say, the critical radius is not necessarily the beginning of chaos. And we definitely have more than one local minima. I do know as I watch these videos (I'm working on N=7 now) they go dim then bright, then dim, then bright almost hypnotically. I guess it's no surprise. It's like the chaos of Radiolarian 7 snapping into order again at Radiolarian 9. Everything comes in cycles.” JS, p6

“Wouldn't N=6.5 produce a the same puzzle as N=13? 7 turns of one side of the N=6.5 puzzle would be the same as 1 turn of the N=13 puzzle. N=irrational always jumbles and I believe N=rational should always reduce to an N=integer case.” Carl H, p6

“So, after 280 posts ... ***ARE*** the Gizmo Gears jumbling? Or have they stopped now? Just curious.” KelvinS, p6

“Yes, that's the question. And the strange way I've seen these behave, I would actually vote that they will be different. Like A'B repeated on a half turn cube might be different from A'B on a normal cube.
We'll see. I'll run a survey of 6.5 and 13 now, and maybe we'll have something to look at this evening.” JS, p6

“Haha. Yes -- though the grammar there always bugged me! Gizmo Gears does indeed jumble.” BH, p6

"…So rearranging slightly, those expressions are actually even more similar:
For $N = 5$, $R = \mathrm{sqrt}((7 + \mathrm{sqrt}(5)) / 2)$
For $N = 10$, $R = \mathrm{sqrt}((7 - \mathrm{sqrt}(5)) / 2)$" BH, p6

Brandon Enright wrote: "The most troubling thing here is that there seem to be small ranges of R greater than the critical radius where jumbling goes away."

Thu Jul 23, 2015 11:48 am "This addresses the first 'if', so that at least seems possible. Is the critical radius of the entire puzzle guaranteed to be the same as the critical radius of the generator with the lowest critical radius?" CH

"Is that even possible? Can a puzzle transition from jumbling to doctrinaire with increasing R? If so that transition may be just as fascinating as the transitions to jumbling that we are talking about. I just don't see how that could even be possible. … ", CH

"Are the Gizmo Gears jumbling? thread is right up there as one of my favorite threads of all time... On any forum. Not only is this a fascinating theoretical discussion but it's so pretty to look at as well. It's a win-win on so many levels. Gizmo Gears took a break there for a while but he is back to jumbling at full speed again." CH

" …in general, just because we find that a particular generator starts jumbling at a given radius, that doesn't mean another one doesn't jumble earlier." BH

"Is the critical radius of the entire puzzle guaranteed to be the same as the critical radius of the generator with the lowest critical radius?" CH

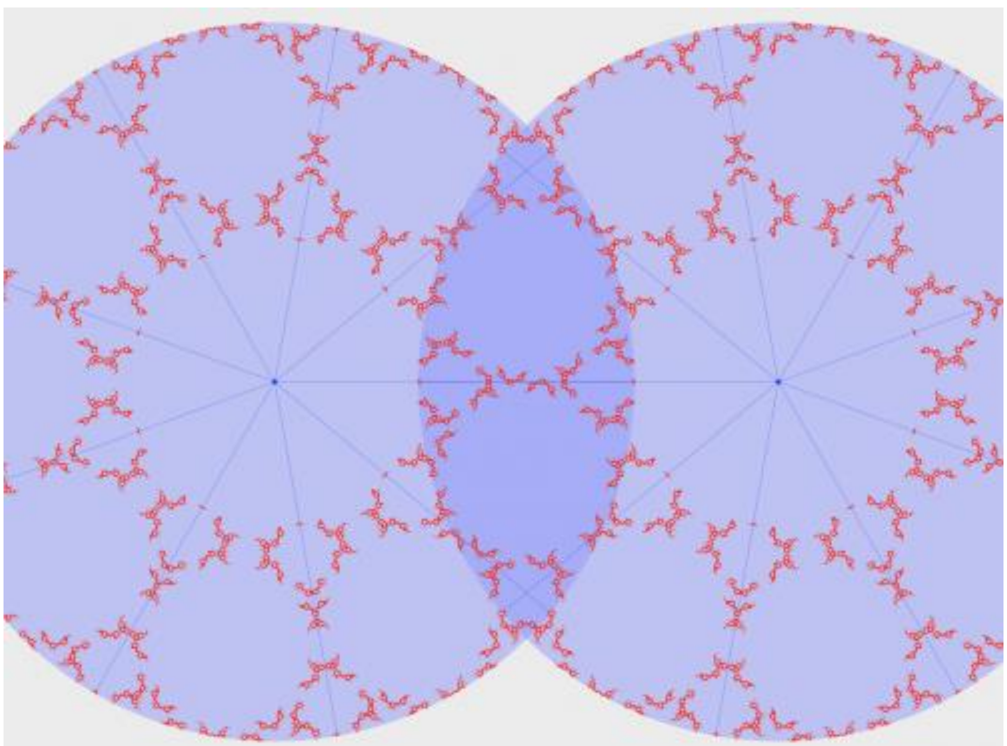
"Well that's a very important question. At this point the answer appears to be yes, but we don't really know. Can there be jumbling without there being a single jumbling generator?" BH

Brandon Enright wrote: "The most troubling thing here is that there seem to be small ranges of R greater than the critical radius where jumbling goes away."

"Is that even possible? Can a puzzle transition from jumbling to doctrinaire with increasing R?" CH

No. For the puzzle as a whole, this is impossible. Suppose we look at a jumbling instance. Then there is a point on a disk perimeter with an infinite image, even if we only use moves for which it never leaves the disk intersection. The reason is that, unlike in the fixed-generator case, leaving the intersection never enables it to reach places (in the intersection) it couldn't anyway. When it eventually moves back into the intersection, it will be somewhere reachable by staying within the intersection in the first place. So, if we increase R, the image of our original point is still at least as large as it was at the original R. In particular, if this image is infinite, it stays infinite." BH, p6

"N=9 does not seem to jumble at my previously identified radius of 1.4085 using [A',B]. Presumably there is some other generator that does jumble there? [A',B] starts jumbling around 1.4294:" BH, p6



N=9, R=1.4294, G=[A',B], theta=0.004776124902

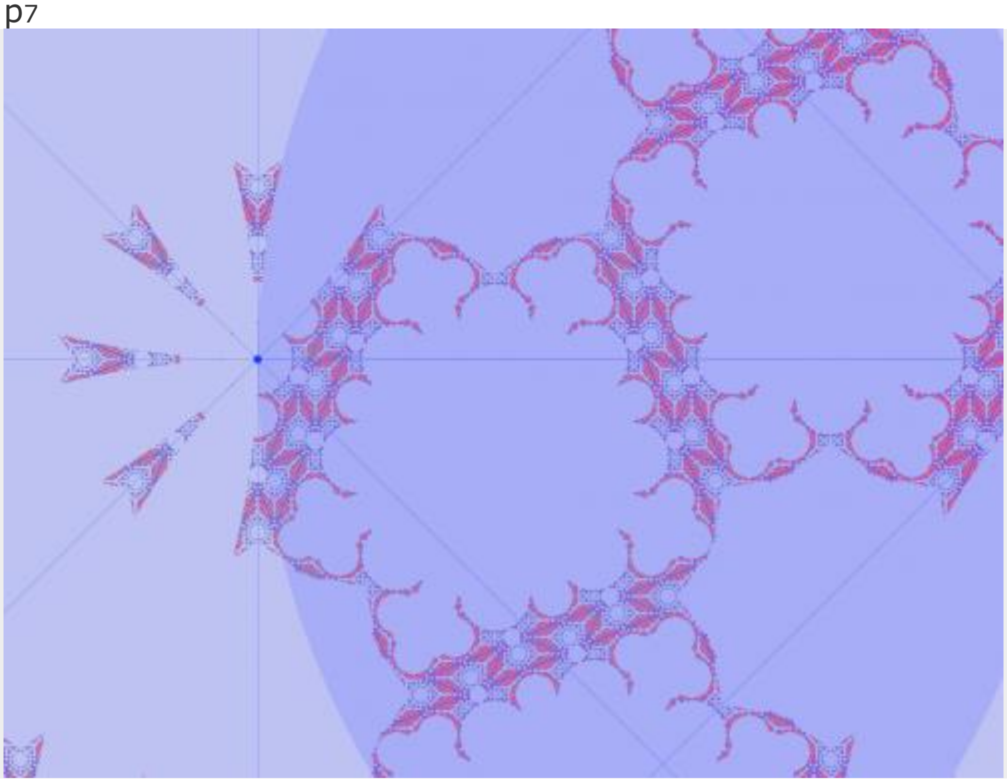
"Jason, I'm with Carl here -- [A',B] for N=2.5 is just [2A',2B] for N=5. So what you've discovered is that [2A',2B] also jumbles for N=5, at the same radius as [A',B]." BH, p6

Page 7 Thursday, July 23, 2015 thru Monday, July 27, 2015

Sorry, guys, I'm taking glances in between tasks at work and it's making me a bit slow.
I see, investigating N=2.5 is just a new sequence on N=5.0. Interesting that if [A'B] jumbles at a point, it's not always true that [2A' 2B] jumbles, right? Because N=10 jumbles at 1.54 with [A'B] but not with [2A'2B] (which is the same as N=5 with [A'B]). Am I on the same page now? If so it means [A'B] and [2A'2B] may generate jumbling together ... or not. Which is interesting in itself, I guess." JS, p7

"Bob, have you surveyed N=8 at 1.742 or thereabouts? Going from 1.741 to 1.742 feels like a transition to me." JS, p7

"No... I can fully resolve N=8, [A',B], even at 1.744. In fact it appears to me that [A',B] doesn't start jumbling until about R=2 (and apparently not 1.99!). Funny, I know I was looking at N=8 yesterday. Either I was looking at a different generator, or I was in a small subrange where it jumbles. It doesn't appear to jumble at the overall critical radius of about 1.712." BH, p7



N=8, R=2, G=[A',B], theta= 0.09688507002

“Here's a survey for N=8 from Radius 1-3. And a zoom in on the area that seems to go crazy for me.
Bob, I'm curious about why it's not looking the same to you. Need to think about this.” JS, p7
http://twistypuzzles.com/forum/download/file.php?id=49146&mode=view

http://twistypuzzles.com/forum/download/file.php?id=49145&mode=view

“Yes, the notation will change but that should be it A'B on N=6.5 should be the exact same thing as 11A2B on N=13. A'B on N=13 will be 6A7B on N=6.5. For any rational N just write it as a reduced fraction and the resulting puzzle should be identical to N=numerator. The only difference will be in the definition of the operations.” CH repeated here on p7

“OK, I'm catching up! Carl, part of the confusion here is that I am not doing full unbandaging’s, just running specific sequences. So, I was interested in whether the emerging patterns would look different, and they do. But, if I was fully unbandaging, then I'd agree they should look the same. And I agree that in effect I'm just asking this question about a different generator.”

“The problem is, no matter where the point is on a disc, a local vector gets the same rotation. For example, on N=12, each point gets +30 degrees or -30 degrees, so it's not a way to reveal irrationality. I hope I'm not misunderstanding your idea. I'm not sure what you mean about tracking the angle using the center. Maybe I'm missing something.” JS, p7

“Indeed my initial thinking before I started the 45 Degree Rubik's Cube was that showing irrational rotation was the way to go. When I did the 45 3x3x3 though I realized that the rotation other than 45 degrees comes from rotating about non-parallel axes. For all 2D puzzles the axes of rotation are always parallel and so pieces, no matter where they're located, always get rotated by however much you twisted they disk they're in.
So for N=12 all points / pieces always have a rotation that's a multiple of 30 degrees.
So the jumbling is in the translation of points, not the rotation of them.
Finding what causes the jumbling through the translation of points is proving to be quite challenging (in the generic case at least).” BE, p7

(Comment by Doug Engel I believe that the irrationality comes from the fact that (a point) in going from A circle to B circle then the center of the cutting circle (cutting circle is the A or B centered circle that makes the new cut) gets moved and becomes a non-crystallographic (irrational) point relative to either or both A and B circles. Thus you can have an unbandaging somewhere for a simple repetition of rotations but if you make random changes in the rotations you should always get jumbling >=Rc. For instance try 20(A, B), 3(A', B) as a generator. With 20 and 3 being mutually prime.)

"1:50am here and I have an interview tomorrow (actually now it's today) at 8am so I'll keep this short. I haven't looked at any math, but I was thinking the rotation about one disc might have an odd effect on the angle of rotation relative to the opposite disc. Though since the two axes are parallel maybe not. Though that does raise an interesting question. In some of these animations it appears that some of the finite sized pieces tend to be circular or at least polygons of much higher order then N. If no piece can have more than N valid orientations why would one see shapes with more symmetry then that? I was thinking the circles could be rotated arbitrarily? If not I'm not sure I understand why they are circles." CH p7

"What makes a generator sufficient for a full unbandage for puzzles with two discs like this? What's the minimum generator, run as many times as you like, that unbandages N fully?

Here are some of the movies I've been promising. Full surveys from R1 to 3. N=3 is pretty uneventful, but I'm including it" JS, p7
http://twistypuzzles.com/forum/download/file.php?id=49156&mode=view
N=5 is awesome!
http://twistypuzzles.com/forum/download/file.php?id=49155&mode=view
N=3 is pretty uneventful, but I'm including it anyway
http://twistypuzzles.com/forum/download/file.php?id=49154&mode=view
N=7 is awesome!
http://twistypuzzles.com/forum/download/file.php?id=49153&mode=view

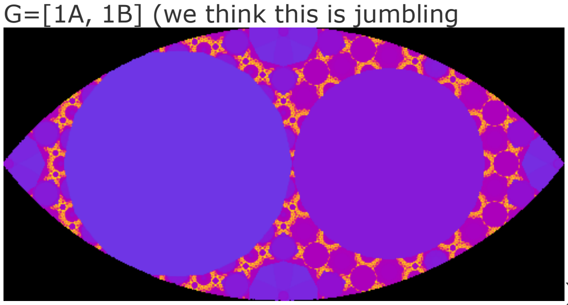
JasonSmith wrote: "What makes a generator sufficient for a full unbandage for puzzles with two discs like this? What's the minimum generator, run as many times as you like, that unbandages N fully?"

Is an infinite unbandage possible with a finite sequence applied infinitely many times? I think no, but I have no justification. Perhaps unbandaging is like a search, and you have to visit every node in the tree, making every possible move." JS, p7

"In general I don't think any single generator will suffice to fully unbandage a puzzle.
An infinite unbandage from a finite sequence applied infinitely many times is basically what we are doing here with our short generator... well not infinitely many times, but we believe we are seeing jumbling, i.e. infinite unbandaging. But we're still not fully unbandaging things by using a single generator.
Yes, fully unbandaging is like search. You have to try every possible move sequence. Fully unbandaged means that from any reachable configuration any move is possible. Or more simply, the puzzle looks the same after making any of the legal moves. This is not the case when just following a generator. The images we get then are closed under application of the generator sequence as a whole, but not after any one single move.
So the reason to look at generators in the first place, even though they don't fully unbandage puzzles, is that they seem to jumble nonetheless, and seem to be behind the jumbling that happens for full unbandaging." BH,p7

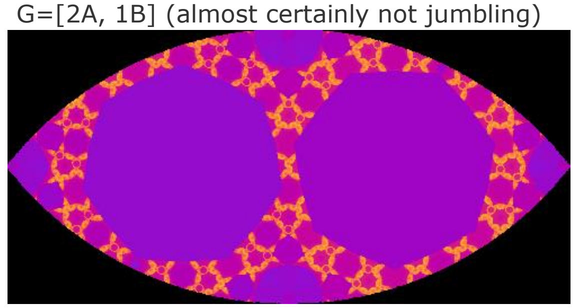
"I did a survey of generators for N=7 at the critical radius R = 1.623579" BE, p7

G=[1A, 1B] (we think this is jumbling

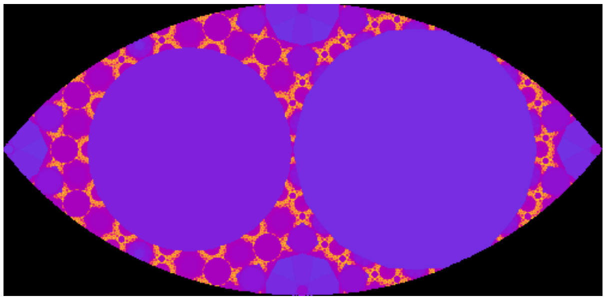


)

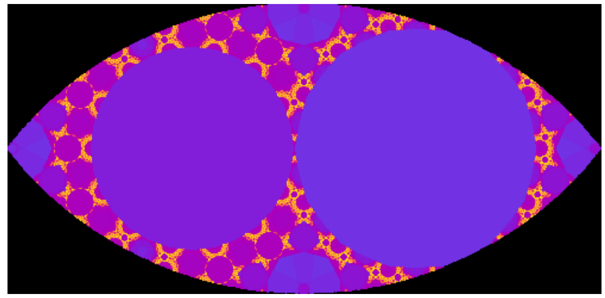
G=[2A, 1B] (almost certainly not jumbling)



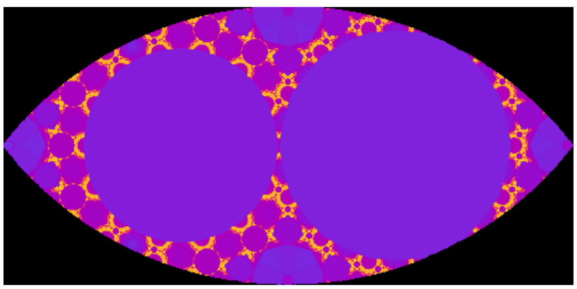
G=[2A, 2B] (we think this is jumbling)



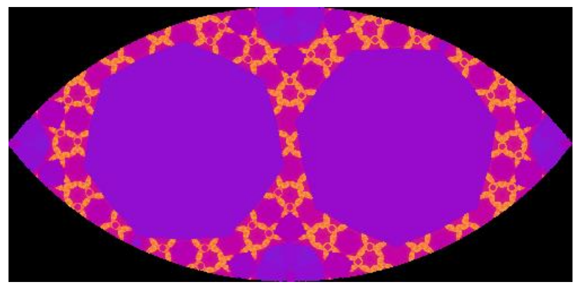
G=[3A, 1B] (almost certainly not jumbling)



G=[3A, 2B] (we think this is jumbling)



G=[3A, 3B] (we think this is jumbling)

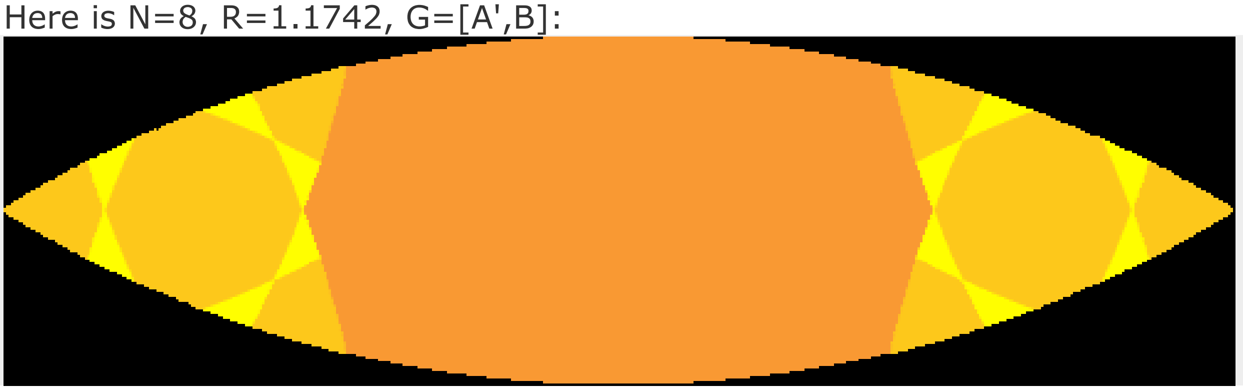


“Well that's funny... the ones where you say "almost certainly not jumbling" are exactly the ones that look a lot like the N=7 fractal. Hmm!” BH, p7

BHearn wrote: “I don't know... here is N=8, R=1.1742, G=[A',B], zooming from the complete set into the highest level of detail there is. It's fully resolved for me.”

“I looked at that generator and got something totally different. Then I realized you meant R = 1.742.” BE, p7

Here is N=8, R=1.1742, G=[A',B]:



"… radius surveys for N=8, N=9, N=10, N=11, N=12 and N=13." JS p7

http://twistypuzzles.com/forum/download/file.php?id=49181&mode=view

http://twistypuzzles.com/forum/download/file.php?id=49180&mode=view

http://twistypuzzles.com/forum/download/file.php?id=49179&mode=view

http://twistypuzzles.com/forum/download/file.php?id=49178&mode=view

http://twistypuzzles.com/forum/download/file.php?id=49176&mode=view

Brandon Enright wrote: "Well I think we're taking a very engineering-like approach to labeling something jumbling. If we can find points that seem to have huge orders we just assume they're infinite."

"When I was building my table of critical radii, it was a little more complicated than that, at least to get high precision. When you're very close to the transition, but low, there can still be points with orders in the millions, and you can't search them all. Right at the transition there is a very sharp fractal pattern. Just past it, the pattern starts to blur. But every extra digit of precision costs you a lot more in search power to detect this.

Also you only see this blurring when looking at the full unbandaging, not the image of a single generator. Though I guess you do get something similar in at least some cases: at N=5, the [A',B] generator pattern is a sharp line right at the transition.

Now I think we are not so interested in numerically finding the radii very precisely, and more interested in understanding the phenomenon. If we can get close, we hope to be able to see a pattern and use geometric reasoning to find the exact radius -- though so far this only works when we have straight lines." BH, p7

"Apart from the problem of proving chaotic dynamics, one thing I'd like to understand better is this notion that a generator can go from jumbling to non-jumbling. Can either of you give me an example of where you think this might happen? Then I can investigate it more fully with my program. If this happens, it must be the case that some other generator has started jumbling by then, which implies a curious kind of relationship between distinct generators.
So ignore what I said above about us no longer being interested in finding critical radii so precisely -- now there is even more work to do, because we want to find transition radii, on and off, for multiple generators for each N.

Since [A',B] doesn't seem to jumble for N=7, but others do, maybe that's the place to start. Does one of those other generators stop jumbling at some point?” BH, p7

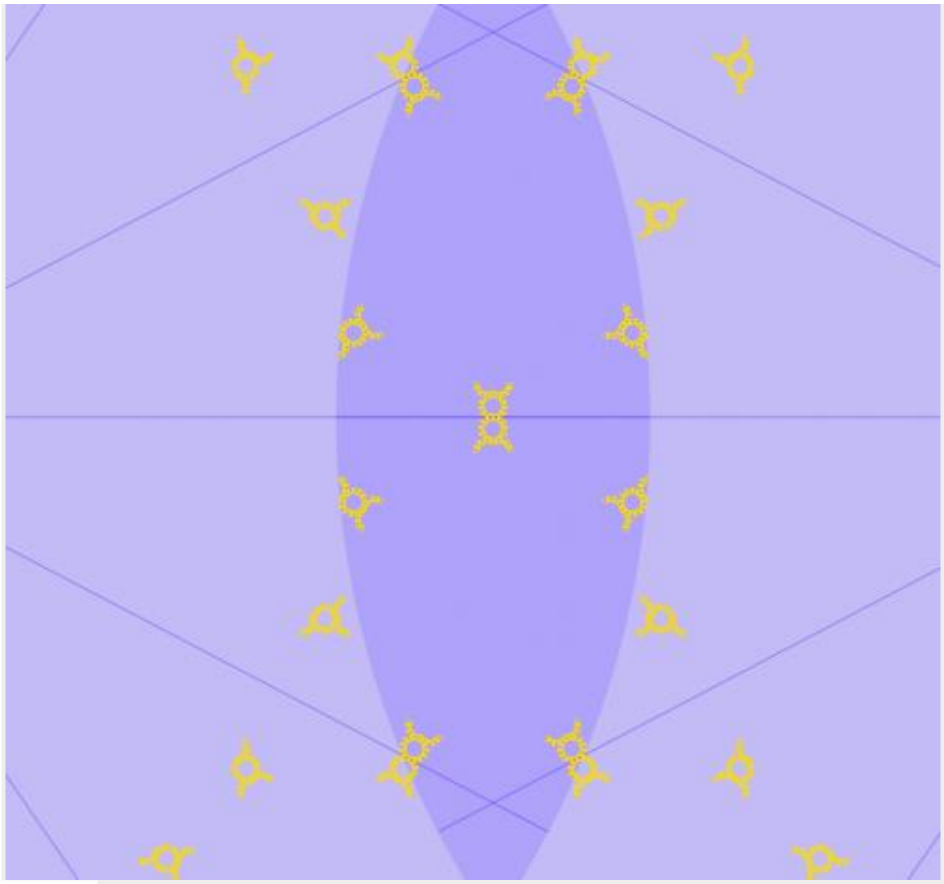
“The movies are very cool, but for me at least I don't see how they help resolve anything quantitatively... maybe I am failing to interpret what they are displaying? Anyway, if you added some info like radius and maximum order to each image, that would help.” BH, p7

“I think they help us see how the order of the pieces is changing as radius changes. Currently pale yellow is above threshold. I would have assumed seeing the order of the pieces changing with more resolution and higher threshold would help us detect where things go crazy and where they calm down again. I do have some font code (writing to the screen canvas of course, and not my output image). I'll see if I can work that out. Unless you guys really don't think it's helping narrow anything down?” JS, p7

“For me, I think maybe they could help quite a bit if I could see radius and maximum order (or threshold if over) on each slide.” BH, p7

“Great, working on that now.” JS, p7

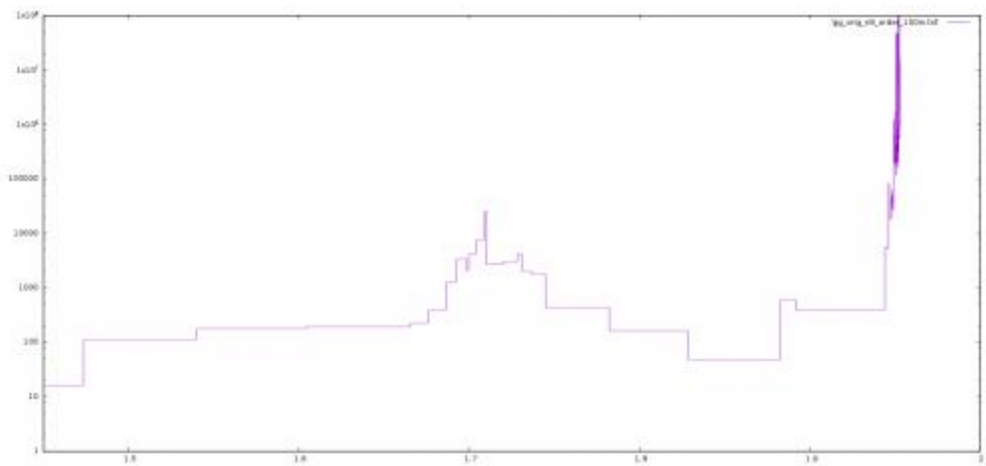
“So here is N=13 at about the critical radius, R=1.21398, with G=[A',B]. This is not straight lines, but it's a relatively simple fractal-type pattern. I've been trying to find some geometry in it that only works at this radius... not succeeding so far.” BH, p7


theta=0.06745062943, 30,000,000 points

"Wow... some absolutely fantastic research here! This should really be part of some sort of lecture during a puzzle event. Who would have thought that a relatively innocent feature could "expand" in less dimensions through a fractal domain? It is exactly what someone wants to see in this forum. 🙂"
Pentazis K, p7

"Well I have given a couple of talks about it. IPP would have been a good venue, I guess. And I was about to get around to writing a paper... actually three, for different journals. But now that Brandon has discovered that simple generators actually do interesting things, everything is different! I think we have a lot to learn yet, and a lot of active avenues of exploration now." BH, p7

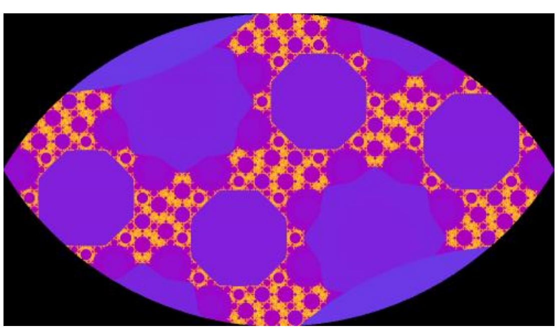
"I finished writing my automation code. Now I can generate images without constant babysitting of the process. I started looking at N=8 where we don't yet have any critical radius estimate. Based on the rest of the data we've gathered it seems like finding a spot where the origin appears to jumble will be very close to the critical radius for a given generator.
So I made some code to measure the order of the origin as R is increased.
Here is a plot for N=8, G=[A' B]:



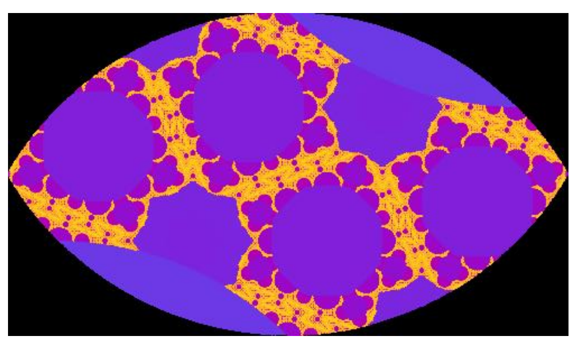
Here is the raw data for that plot: **gg_orig_n8_order_100m.txt** (1.5 MiB)
Downloaded 1 time

Note that the cutoff is 100 million which is first hit by R=1.95199 (my R step size is 0.00001).
The first R my program found where the origin's order is > 10m is 1.95032 and at R = 1.95035 the order of the origin is ~48m.

Here is a render of N=8, R=1.95032, G=[A', B]:



When I increase R to 2 this is what I get:



It seems pretty likely to me that G=[A', B] jumbles before R=2." BE, p7

JasonSmith wrote: "Brandon, that image is fantastic. Is your max 100,000?"

"Sorry I missed your question. My max was 200 million. A lot of points were above the 200M cutoff too. It's possible not all variations behave the same but on all the previous puzzles I've explored when you're really close to the critical radius for a generator a very small percentage of the randomly selected points have orders this high. That's one of the reasons I think R=2 is well above the critical radius for the generator. EDIT: I should mention that I don't color a pixel at all (set it to white) if I can't get a sample for it. So I think your code just saturates pixels (with yellow) that are at your threshold whereas my code makes a distinction between knowing the order and the order being above the cutoff threshold." BE, p7

Brandon Enright wrote:"I rendered a **really big** version of N=8, R=2, G[A', B]. …"

"Compare to my version of the same thing, near the top of this page. Note that in mine, all colored points are over my threshold: it's all the image of a single point on the perimeter. I didn't record what the threshold was, however. That image is colored by depth (number of moves to reach it from the starting point)." BH, p7

Brandon Enright wrote:" I started looking at N=8 where we don't yet have any critical radius estimate. Based on the rest of the data we've gathered it seems like finding a spot where the origin appears to jumble will be very close to the critical radius for a given generator."

"You mean, we don't have a critical radius estimate for G = [A',B]? We do have an overall critical radius estimate, in my spreadsheet, 1.711 - 1.712. I don't understand why you think the behavior at the origin should be indicative of where jumbling starts. There's nothing special about the origin, in general. There's also nothing really special about theta = 0 on the disk perimeter, though I'd think that might be a more useful point to track." BH, p7

JasonSmith wrote: "I feel like we're definitely at a point now where we can say that with a specific generator, jumbling comes and goes as radius increases. A very interesting result, unless I have missed something."

"I agree, but I don't see why that would follow from Brandon's plots. But from your N=5 movie, it definitely appeared to be the case. Based on that, I looked specifically at R = 2.6, and I can fully resolve that. So that at least appears to be one very clear case where jumbling starts, and then stops, at least for a given generator. We know that this is impossible for the overall puzzle. So it would be nice to start nailing down the transition ranges for each generator. Which generator jumbles at 2.6? If none do, then we have overall jumbling without any single generator jumbling." BH, p7

Page 8 Monday, July 27 2015 thru Saturday, August 21 2015

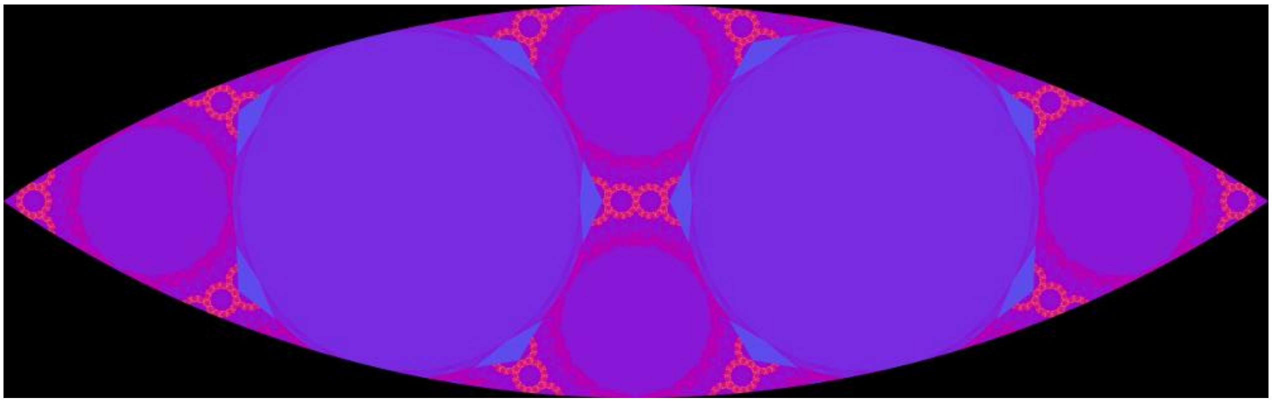
BHearn wrote: "I don't understand why you think the behavior at the origin should be indicative of where jumbling starts. There's nothing special about the origin, in general. There's also nothing really special about theta = 0 on the disk perimeter, though I'd think that might be a more useful point to track."

"In all of the critical radius jumbling we've looked at, the origin has either been part of the image for one of the infinite edge arcs or been very close (and pretty high order) to the infinite image points.
So I don't think there is something special about (0, 0) where it will tell us the critical radius exactly. I just think that it tends to be located quite close to the infinite image points so I think it serves as a good canary. Also, we've seen that slightly above the critical radius the infinite image seems to get "smeared out" and even closer (or contain) the origin. So if the origin order is very high (or seemingly infinite) we're probably close to the critical radius." BE, p8

Twisty Renderer -- Jaap's Spheres with POV-Ray

Bob Hearn wrote: “So here is N=13 at about the critical radius, R=1.21398, with G=[A',B]. This is not straight lines, but it's a relatively simple fractal-type pattern. I've been trying to find some geometry in it that only works at this radius... not succeeding so far.”

“I made a really big rendering of this. Click for the full size:” BE, p8



“Funny that you guys were talking about behavior at the origin. I was just finishing these critical zooms for posting. These ran for days. The max iterations are animated to increase with the zoom and it happens fast!
Each zooms in on the origin with the puzzle set to the critical radius. I have N=2.5 (also known as N=5 using [2A' 2B]), N=5, N= 10 and N=12.
N=12 is at sqrt(2) instead of the critical radius, to reveal the snowflake.” JS, p8

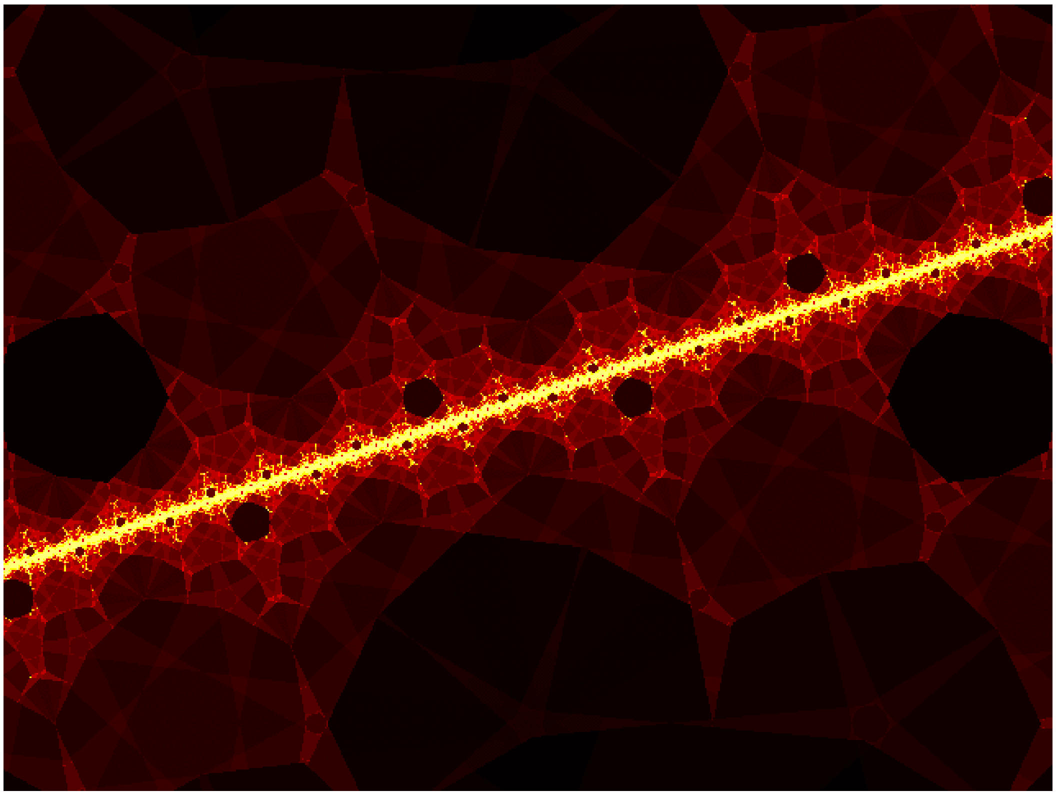
Jason Smith posted here as 'io' through 2012.
Visit Jason Smith's PuzzleForge on Shapeways!
Jason Smith's Puzzles - YouTube Channel.
Check out my renderer DeluxeRender (written from scratch).

"Those are so beautiful! I'm really partial to N=10 but the snowflake curve for N=12 is hard to resist too . I'm counting on you somehow working these into a big blockbuster movie! Imagine a superhero that casts N=12 snowflake fractal lightening bolts or a N=10 zoom wormhole sequence... For N=12, is that actually at the critical radius or is it at R=sqrt(2)? I thought the snowflake didn't show up at the critical radius." BE, p8
Twisty Renderer -- Jaap's Spheres with POV-Ray

"Oops! Yes, it's at sqrt(2). Edited above.
By the way, I have implemented a 9x-12x speedup in my code for the special case of normal surveys using A'B which is working well. I'm very excited about that. I'll be able to run hires surveys quickly, so please let me know if you want a survey somewhere!" JS, p8

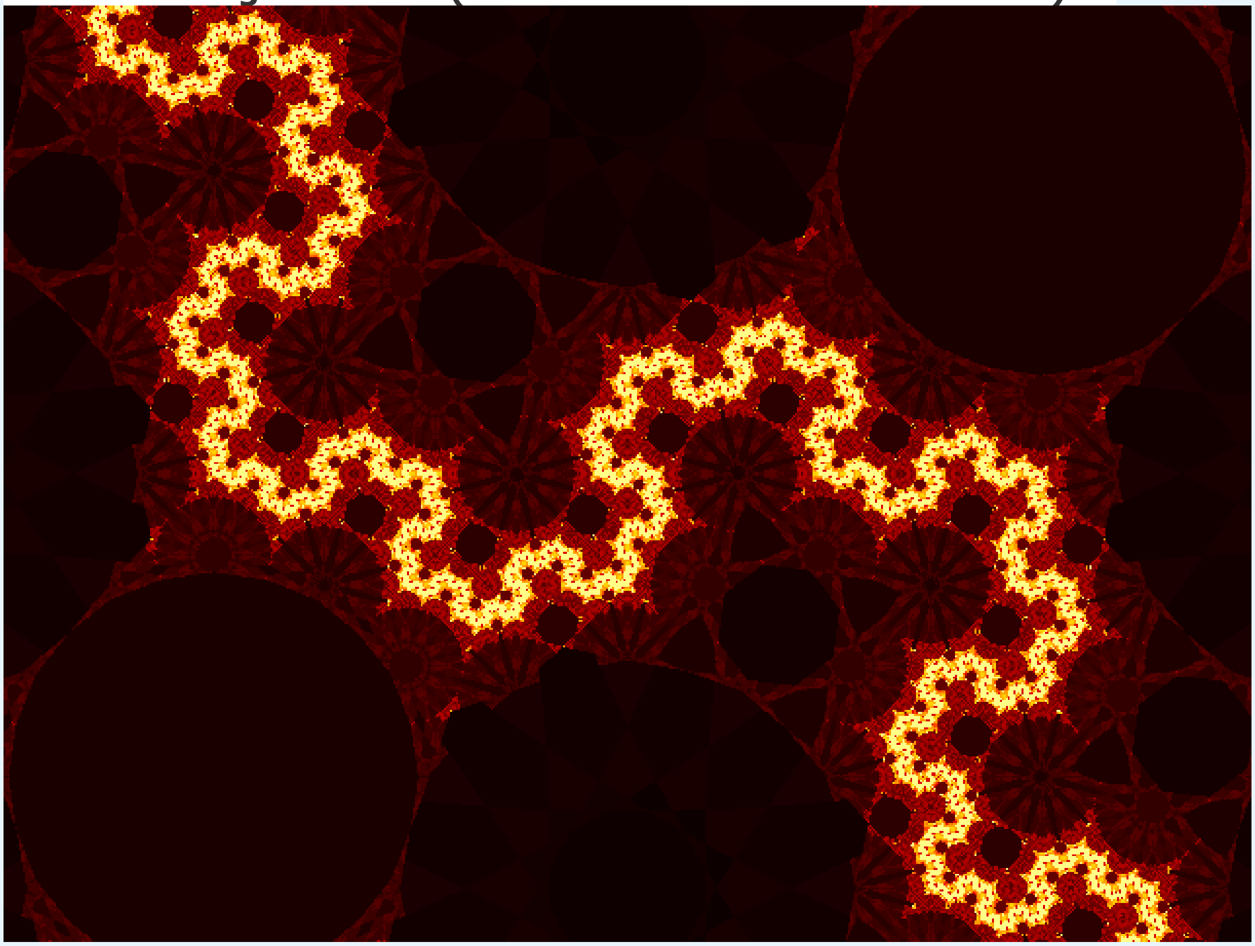
"Sorry guys, I've been mostly out of it for a couple days, tough 50K on Sunday, just catching up.
So we now think [A',B] does jumble at the critical radius for N=8? For some reason I was thinking it didn't, and it would be a different generator. I'll look for this too. BTW I also see jumbling here at R=1.952, though I really had to push my parameters to expose it (theta step of 10^-11). I haven't tried to go lower." BH, p8

“I've also got a survey going for N=8 from 1.7112 to 1.7114 in 200 frames. It's already pretty unstable at 1.71121, hanging around in the millions, but not yet at a full jumble. I think I'm also close to a jumble for N=24 around 1.0711-1.0712. Still searching. Also, I'm running full surveys with my new hires annotated settings for all N up to 24.” JS, p8



“I had some fun with your N=5 zoom. I took it apart, tried to normalized the brightness between frames, and then stitched together a self-similar region into a loop: As you can see it isn't perfect but still looks cool and does pretty accurately show what infinite zoom would look like.” BE, p8
(this plays in a loop on the forum but this shows the general state)DE

Brandon Enright wrote: (this is J Smith's snowflake zoom)DE



"Great! This will be my newscreen saver 🙂" Oskar, p8

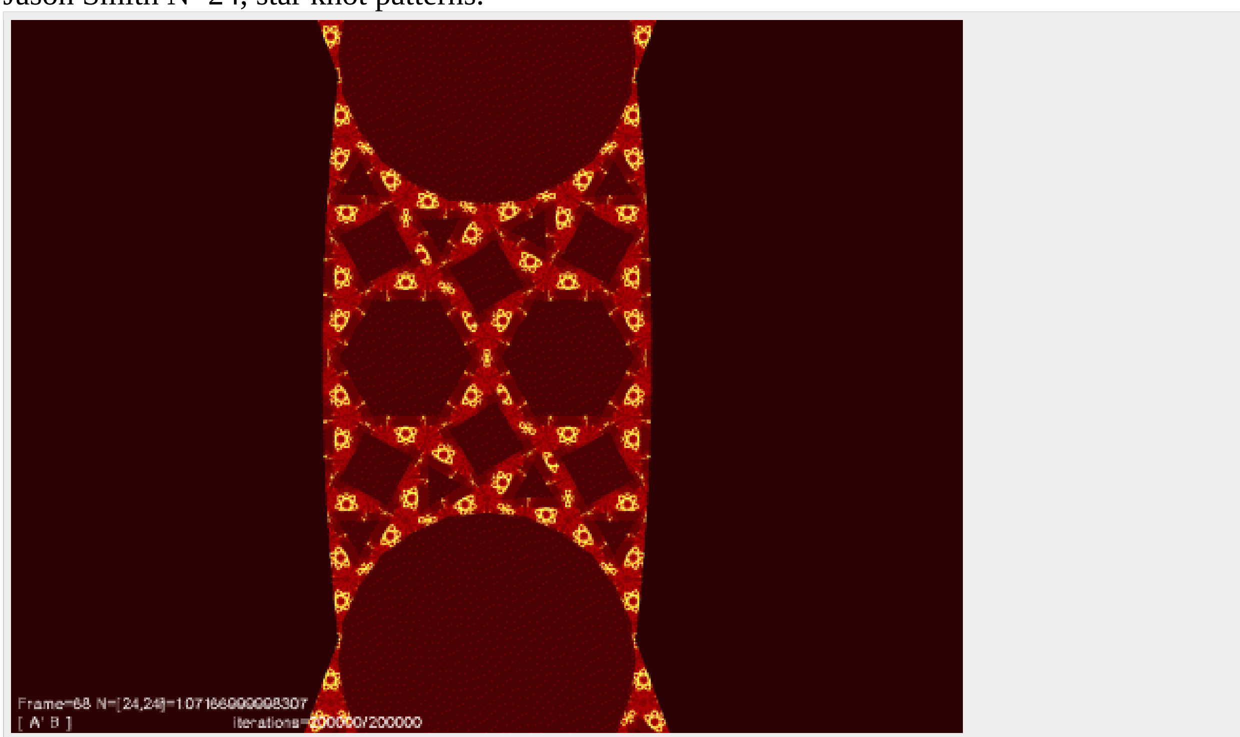
"… Just a quick update from my survey of N=24. I believe I have found a jumbling transition around r=1.07167. It makes these interesting little star-knot patterns. I'm working on a finer survey with 20 million threshold to try to nail it down exactly. … JS, p8

Brandon Enright wrote: "It doesn't work quite as well because there seems to be a difference in detail between the outer copy and the zoomed copy."

"Oh it certainly works well enough... it took me forever to spot the issue you were seeing. I think you are referring to the black octagons that shot out from the center toward the lower left and upper right. In the animation at

the end they suddenly pop into red circles. That and the large dark region next to it losses its interior details. Jason could you continue your zoom in a little further? I believe that would allow Brandon to remove the frames containing this issue and make for a smoother animation. Though to be honest it looks fantastic as is.
Some comments:
(1) As all the cuts are made using an arc of fixed radius, I'm actually surprised these regions are so self similar as they zoom in. What are arcs initially must be approaching straight lines as we zoom in and I certainly can't pick up on that in these animations.
(2) I've seen similar animation of the Mandelbrot set. Which makes me wonder, how was the Mandelbrot set proven to be infinite? Not sure I'd understand it even if I saw the proof but I'm now curious if a similar approach could be used to prove jumbling here." Carl Hoff,p8

Jason Smith N=24, star knot patterns.



We could define an analogous set for these disk puzzles that are all the points in the intersection of the two circles (the wedge) that don't have infinite orders when you apply some generator sequence over and over. The

trouble is that when points leave the wedge they're still in one of the two circles and so moves in the generating sequence for the circle they're not in no longer affect the point.
To implement my simulation I have a conditional if statement checking the distance of the point to the center of the circle before I apply the transformation matrix for that circle to the point. I think its this conditional operation that makes analysis extra hard.

I did some testing for the generator [A, B] assuming points never leave the wedge. When N is even [A, B]x(N / 2) is the identity transform. When N is odd [A, B]xN is the identity transform.

There is probably some simple proof by construction to show this but I just manually checked a bunch of N." BE, p8

JasonSmith wrote: "Brandon, you mean the puzzle is actually unscrambled again? What about with deeper "r" values? Or is that what you mean that points never leave the wedge? That the cuts are shallow enough not to overlap?"

"I mean for any point that never leaves the wedge through the process of [A, B] over and over. If a point leaves the wedge after the third application of [A, B] then you can't apply [A, B] a fourth time because either A or B won't move the point now that it's not in the wedge.
Another way to think about it is that if the circles had infinite radius then all points would always stay in the "wedge" and all points would always cycle back around after N applications of [A, B] when N is odd or N / 2 steps when N is even." BE, p8

"Along the lines of the testing above, I decided to look at the effect of [A', B] (assuming a point stays in the wedge after A' so that B can be applied to it). It should come as no surprise that for any N, the effect of [A' B] causes the rotation to cancel and the transformation be reduced to translation only. I

assumed that the distance a point was from the center of A and B would affect how much it gets translated but it turns out this is not correct. A point will aways get translated by the same amount no matter where it is located.
I will represent how much a point gets moved by (x, y). So point (0, 1) becomes (0 + x, 1 + y) after [A', B].
N = 3 -> (3, tan(60 degrees))
N = 4 -> (2, 2)
N = 5 -> ((5 - sqrt(5)) / 2, sqrt(((1 + sqrt(5)) / 2) + 2))
N = 6 -> (1, tan(60 degrees))
N = 7 -> (2 - (2 * cos((2 * Pi) / 7)), 2 * cos((3 * Pi) / 14))
N = 8 -> (2 - sqrt(2), sqrt(2))
N = 9 -> (2 - 2 * cos((2 * Pi) / 9), 2 * cos((5 * Pi) / 18))
N = 10 -> (1 - ((sqrt(5) - 1) / 2), sqrt((5 / 2) - sqrt(5 / 4)))
N = 11 -> (2 - 2 * cos((2 * Pi) / 11), 2 * cos((7 * Pi) / 22))
N = 12 -> (tan(15 degrees), 1)
I manually brute forced the algebraic numbers. The others are probably algebraic too but I just couldn't find closed form exact representation.
Edit: found the closed form for a few more numbers.

Edit: and more closed form numbers. Google works wonders for algebraic decimal expansions 🙂
Edit: pfew all decimals turned into their closed form. I'm noticing a pattern!"
BE, p8

"I have a hypothesis about what causes the infinite order points / jumbling.
If there are places in the wedge where the number of times a point is turned by A' is the same number of times the point is turned by B then the rotation cancels and all that is left is the irrational translation.
I suspect if we look at the points that jumble (or the ones very close to jumbling) we'll find that the A' turns and B turns are very close.
Put more formally, I suspect points that jumble have the property that lim N -> inf of [A', B]xN for a jumbling point will be (A' turns / B turns) = 1" BE, p8

“So here is the pattern.
For [A', B] the rotation cancels and a point gets translated. The amount a point gets translated is:

x moves by 2 - 2 * cos((2 * Pi) / N)
y moves by 2 * cos(((N - 4) * Pi) / (2 * N))
So here is an example where N = 128:
x moves by 2 - 2*cos(Pi / 64) =
0.002409087589655214570456790481798611113592770590776411314365852628780188095685192337621511467211657918
y moves by 2 * cos((31 * Pi) / 64) =
0.09813534865483602850990995388536531662949072605150584042024510653833187912792151364444153422360144751
2 * cos((m / n) * Pi) is an algebraic integer. Therefore [A', B] will always move points by an algebraic integer amount in x and y. Again, this assumes both A' and B can be applied to a point.” BE, p8

Brandon Enright wrote: “If there are places in the wedge where the number of times a point is turned by A' is the same number of times the point is turned by B then the rotation cancels and all that is left is the irrational translation.”

Brandon Enright wrote: “2 * cos((m / n) * Pi) is an algebraic integer.”

“Are all algebraic integers irrational? I'm going to guess no.” Carl Hoff, p8

“I think there is something special about the origin. Symmetry. The fractals that we've seen that follow a path, like the snowflake for N=5, start at the top point of the wedge and propagate to the bottom point. I believe symmetry dictates that this path must also pass through the origin if the path is continuous. The notion of it just being close bothers me. I would think it would have to be right on top of it. Granted we have seen other types of fractals that aren't continuous but I believe the ones that are must pass through the origin.” Carl Hoff, p8

“Are all algebraic integers irrational? I'm going to guess no.” CH (see above)

"No however the only rational values of cos() for rational angles are {-1, -1/2, 0, 1/2, 1}. I ran into a proof of this (several different times actually) while working on the 45 degree 3x3x3. Since there is always a rational angle in the formula the only values 2 * cos(...) could take are {-2, -1, 0, 1, 2}. In other words, we can enumerate every N value where the X or Y translation value is rational and we can know that every other value for N produces an irrational translation." BE, p8

Carl Hoff wrote: "I think there is something special about the origin. Symmetry. The fractals that we've seen that follow a path, like the snowflake for N=5, start at the top point of the wedge and propagate to the bottom point. I believe symmetry dictates that this path must also pass through the origin if the path is continuous. The notion of it just being close bothers me. I would think it would have to be right on top of it. Granted we have seen other types of fractals that aren't continuous but I believe the ones that are must pass through the origin."

"That's a good point, but only for special R and particular N. Brandon was using the origin as an indicator of jumbling over a range of R, and I still don't buy that." BH, p8

"Hey guys- I've still got a few of my surveys running for N=1 to N=24. But I thought I'd post the data I have collected so far:

Google Spreadsheet Here
For N that already have a theoretical solution, I have only shown the initial measurement to hundredths and using 100,000 iterations, and have not invested time in re-validating them by experiment beyond pretty pictures we have already seen.
For more accurate entries, I'm using 30,000,000 iterations.
My value for N=7 is different from Bob's, but I suspect his very high threshold has made his estimate more accurate. EDIT- Hmm, I'm still at 1.627556 at 300,000,000 iterations. Maybe it jumbles earlier in the full unbandage.
My value for N=9 is different and I don't know why. Mine jumbles later using

A'B so my guess is that the full unbandage jumbles a bit sooner?
EDIT3 - I upgraded my solve to 300,000,000 iterations and it's still different, so I'm assuming that my above guess is true for now.
My value for N=11 is a bit lower starting at decimal place 10^-4. I wonder if my threshold is just flagging a jumble sooner?
EDIT2- tested at 300,000,000 iterations and I'm still different, so either Bob's is flagging a little early due to low threshold, or something significant is going on.

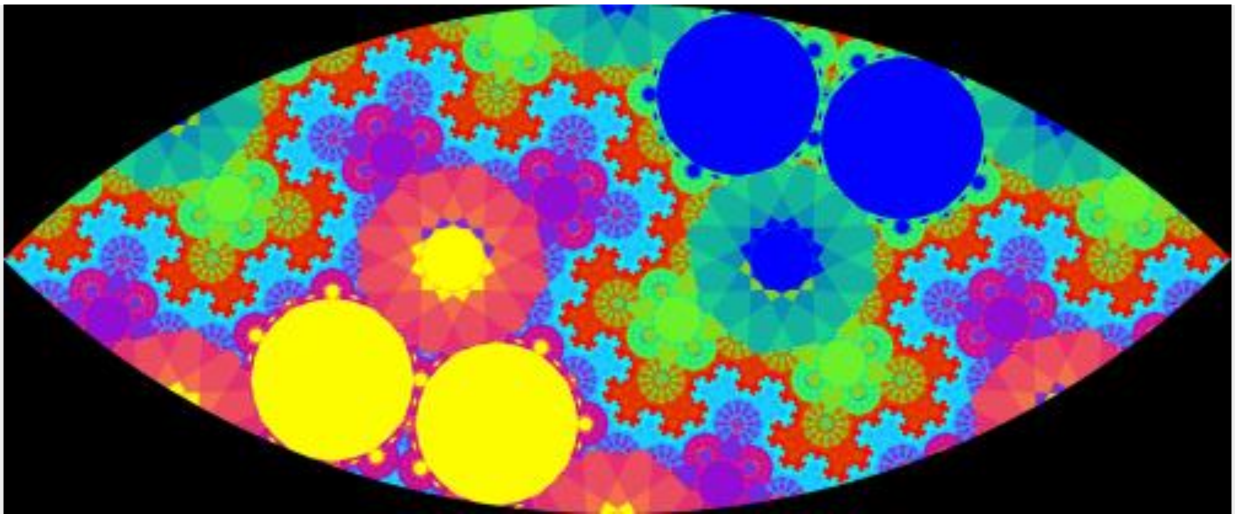
Other values are remarkably similar, which is interesting since mine are A'B, not a full unjumble. I should have the rest of the data tomorrow sometime. I also have 15 gigs of hires images/movies from these jobs. I'm not sure the best way to upload them, but I want to be a little more organized about it." JS, p8

Brandon Enright wrote: **"**I have a hypothesis about what causes the infinite order points / jumbling. If there are places in the wedge where the number of times a point is turned by A' is the same number of times the point is turned by B then the rotation cancels and all that is left is the irrational translation.
I suspect if we look at the points that jumble (or the ones very close to jumbling) we'll find that the A' turns and B turns are very close.
Put more formally, I suspect points that jumble have the property that lim N -> inf of [A', B]xN for a jumbling point will be (A' turns / B turns) = 1"

"I looked at this more. It seem to be true for N=12, R=sqrt(2), G=[A', B]
Along the lines of what Jason has already done, I recorded the number of times A' was applied as well as B before a point cycled back to it's original spot. The highest ratio of A to B seems to be 11:1 and the lowest ratio seems to be 1:11. There is a 180 degree symmetry about the origin inverts the ratio. So if you have a point with a ratio of 2:1 then the other point after the 180 degree rotation will have a ratio of 1:2.

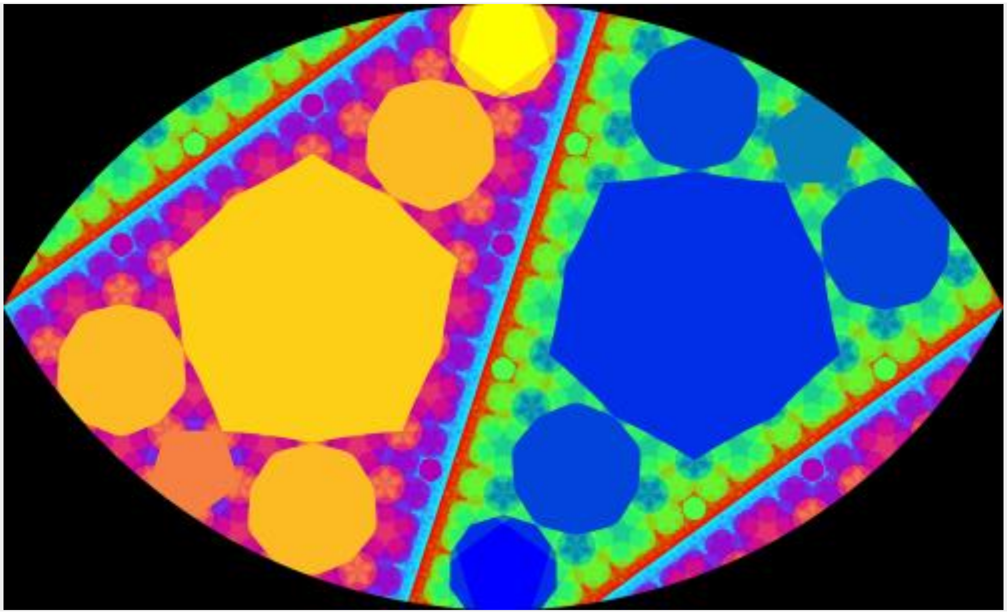
The closer you get to the snowflake curve the closer and closer A and B become. The ratio approaches 1. To plot this I assigned one set of colors to ratios in the range (1, inf) and an inverted set of colors to ratios in the range (0, 1). This results in a sharp transition right at the center of the snowflake curve from one color mapping to the other: Finding a way to map colors in over this domain ended up being a bit of a challenge.

I think the important take-away here is that the closer you get to the snowflake curve the closer A' and B get and since on average their rotations cancel all that's left is the irrational translation in x (which is 2 - sqrt(3)) and the rational translation in y (which is 1).” BE, p8



“Jason, what's your comment in the spreadsheet about N8 possibly not being 'true' jumbling mean?
An interesting question I've wondered about for a while is whether the variant of a dino cube where the corners can turn 60 degrees instead of 120 jumbles (assume a shape mod of the whole thing to a sphere).
There are also of course interesting questions about what happens when the two radii are different, and when the amounts the degree of rotation the two circles do with each move are different.” Bram Cohen, p8

“No surprise, the jumbling boundary for N=5, R=sqrt((7 + sqrt(5)) / 2), G=[A', B] is the place where A' and B are balanced. This causes the rotation of points to cancel and translations by ((5 - sqrt(5)) / 2, sqrt(((1 + sqrt(5)) / 2) + 2)) to be left. That's a total translation (Pythagorean theorem) of sqrt(10 - 2*sqrt(5)):” BE, p8



"Bringing things full circle I believe you can easily spot those yellow points in the very first image I ever posted in this thread back on page 1." Carl Hoff, p8
viewtopic.php?p=303652#p303652

Brandon Enright wrote: "This results in a sharp transition right at the center of the snowflake curve from one color mapping to the other:
Finding a way to map colors in over this domain ended up being a bit of a challenge."

"Well you did a brilliant job. That is simply BEAUTIFUL!!! Can you render this with the two full circles showing as well? The higher the resolution the better... or better yet send me something that allows me to render it so I don't slow down your work. I'm highly tempted to get that printed and framed to hang up in my living room. By the way, I see Douglas A. Engel is listed as the inventor of Gizmo Gears. Anyone know if he is aware of this thread? If not I bet he'd enjoy taking a peak but I'm not certain how to contact him." Carl Hoff, p8

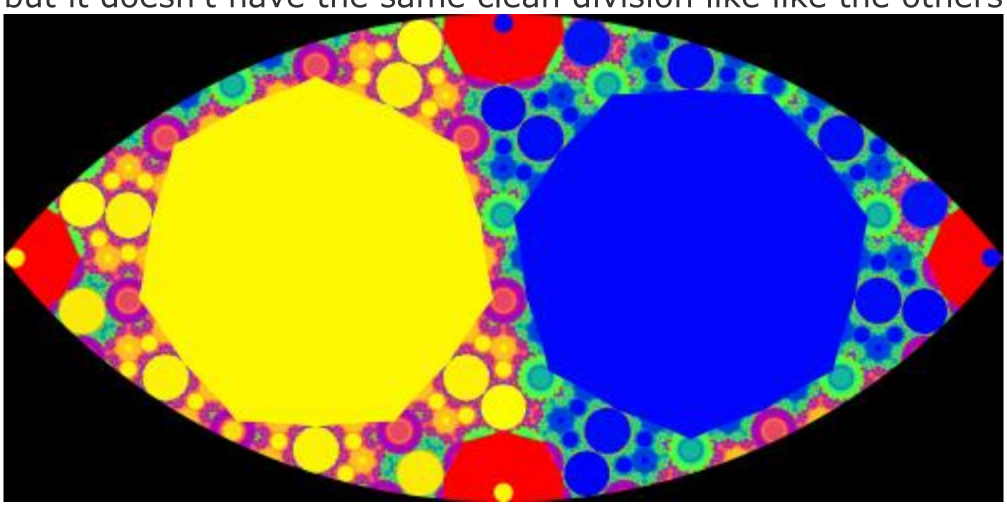
"Yes, he's posted in this thread, several pages back." BH, p8

“Carl, as Brandon mentioned, my program can visualize this way as well, and I'd be happy to save a super hires image, but I'd need Brandon's color mapping. I can imagine it was not easy to come up with. I tried B, G, R, C, M, Y without it coming out as pretty. Or I can give you my executable, since the color mapping is part of the input file!” JS, p8

Brandon Enright wrote: “This results in a sharp transition right at the center of the snowflake curve from one color mapping to the other: Finding a way to map colors in over this domain ended up being a bit of a challenge.”

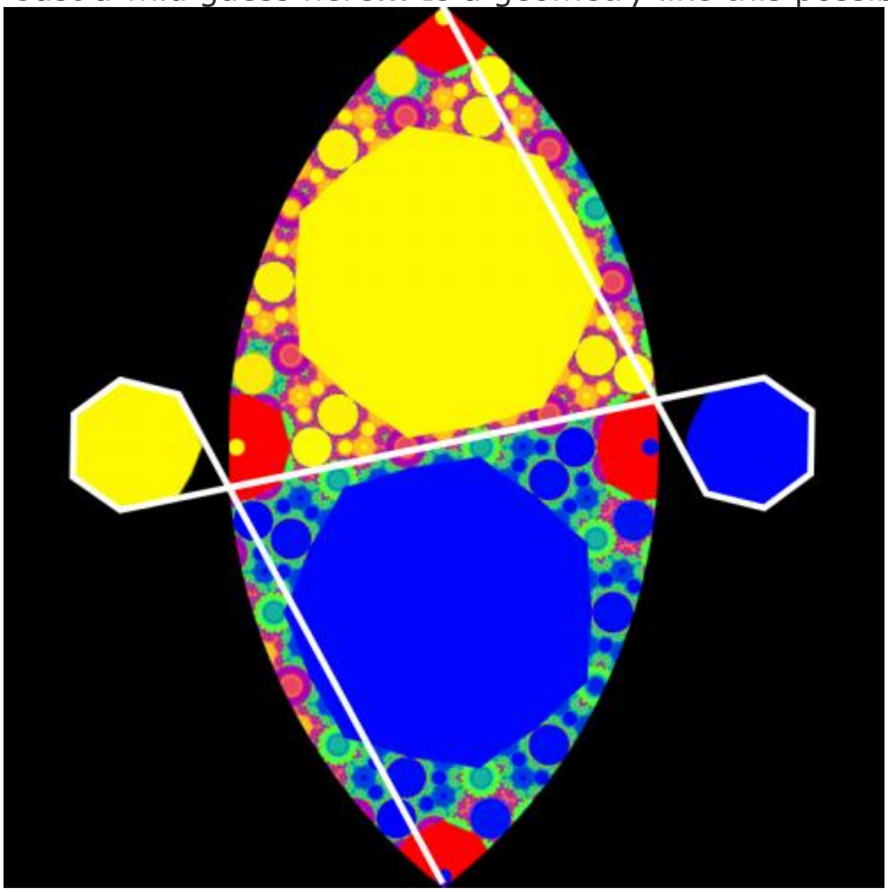
“And yes, this is very beautiful. I've had many people tell me I should produce some artwork based on earlier images from my program -- and a request for a submission to the Journal of Mathematics and the Arts -- but the ones with filled area are much prettier.” BH, p8

“N=7 seems to be our biggest troublemaker because it doesn't match the same jumbling pattern we've seen on others. Here it is at R=1.628 which seems to be sufficiently deep enough for G=[A', B] to jumble. All of the points that jumble here do seem to have a perfect balance between A' and B but it doesn't have the same clean division like like the others.



The sharp transitions between the two color mappings makes trouble for my anti-aliasing routine so there is a lot of red and teal pixel noise right at the transitions.” BE, p8

“Just a wild guess here... Is a geometry like this possible for N=7?



The angle of the central line would be (720 / 7) - 90.0 = 90 / 7 = 12.8571 degrees. The length of the long line segments would be the same. If this is possible, what is the R value that makes it exact?” BE, p8

“Hey Brandon, if you're looking at 1.628, you're close to the radius 1.627556, which is what I measured and have in my spreadsheet as the jumble radius.
Maybe try that?” JS, p8

Page 9 August 1, 2015 thru November 2, 2015

“I'll see if I can work out the trig later tonight, if nobody beats me to it. If it works at all, I'm curious whether we'd get the absolute critical radius,

1.6235789, or the [A',B] jumbling radius, closer to 1.628. I'm guessing the latter, because a geometry like what you drew seems to arise for [A',B].”
BH, p9

“… Here's my math:

B is the origin

A is the center of one of the rotator on the right.

C is the top vertex of the septagon on the right.

G is the septagon vertex that lies on AB

E is the intersection point of the two circles, so that the radius is EA

BA = 1

ACB = (5pi)/14 half the interior angle

BAC = (4pi)/7 septagon geometry

ABC = pi - ACB - BAC

AC = sin(ABC) / sin (ACB) law of sines

AG = AC both are vertex to center distance on the septagon

BG = BA-AG = 1-AG

BGF = (pi - (5pi/14))

BGE = (pi - (pi - (5pi/14)))

EBG = pi/2

BEG = pi - EBG - BGE interior of triangle BGE angles add up to pi

EB = (BG* sin(BGE)) / sin(BEG) law of sines

1^2 + EB^2 = r^2 Pythagorean theorem on ABE

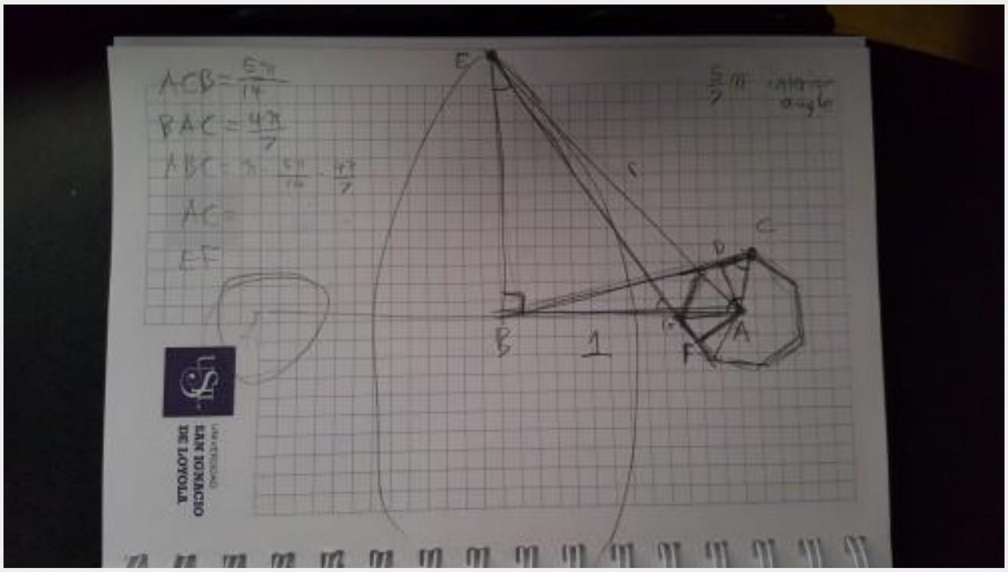
r = sqrt(EB^2 + 1)

r = 1.8560823979318992

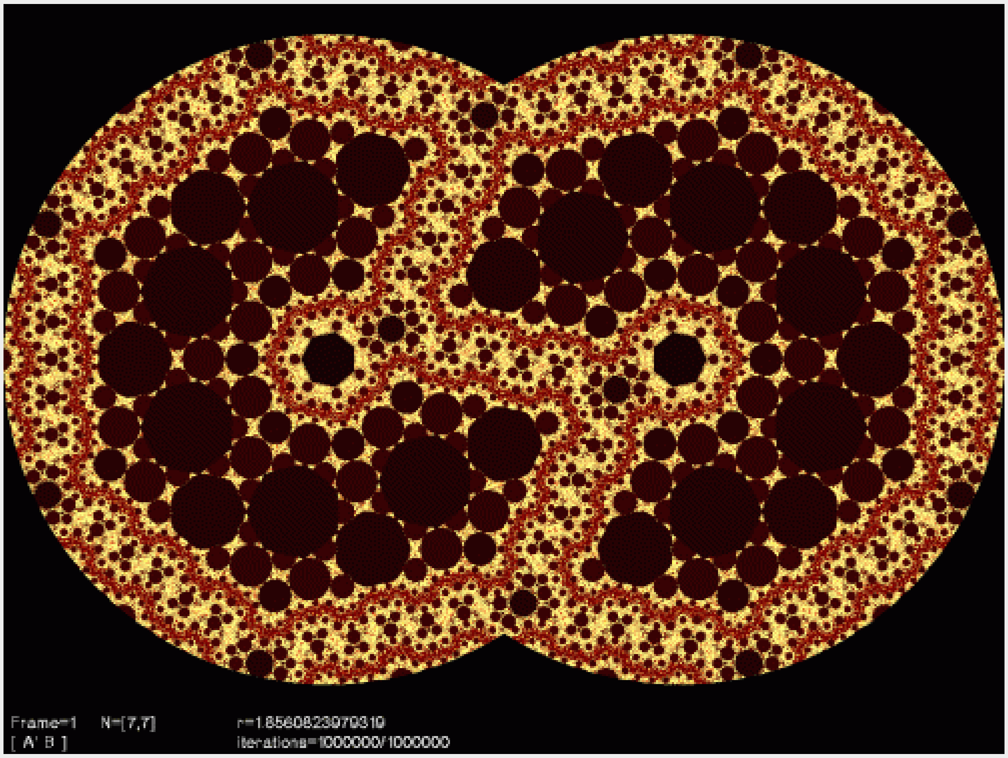
In python code:

```
from math import *
ACB = (5.0 * pi) / 14.0
BAC = (4.0 * pi) / 7.0
ABC = (pi - ACB - BAC)
AC = sin(ABC) / sin(ACB)
BG = 1.0 - AC
BGF = (pi - (5*pi/14.0))
BGE = (pi - BGF)
BEG = pi - ( (pi / 2.0) + (pi - (( 5.0 * pi)/ 14.0)))
```

```
EB = (BG * sin(BGE)) / sin(BEG)
r = sqrt((EB*EB)+1)
print (r)
1.8560823979318992
```



"… Higher res version with 1 million threshold." JS, p9



"Math looks good to me." BH, p9

Google Spreadsheet Here

For N that already have a theoretical solution, I have only shown the initial measurement to hundredths and using 100,000 iterations, and have not invested time in re-validating them by experiment beyond pretty pictures we have already seen. For more accurate entries, I'm using 30,000,000 iterations. The chart is now complete, including values for N=100 and N=1000" JS, p9

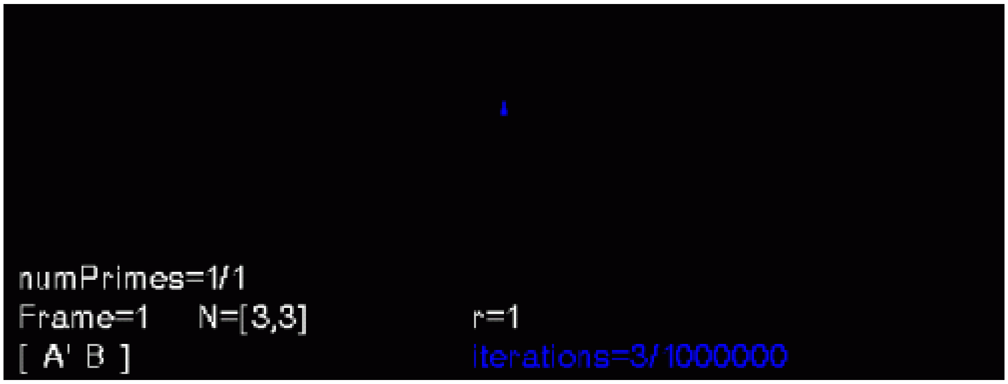
"Here's an N=3 run of 2000 frames, counting primes. Seems like they're 2.7 times more likely than usual. I think I'm a human who saw the pattern 3, 5, 7, 11, 13 and got carried away.
But I've colored the prime pieces blue in the movie. Can you find a pattern that would always give us a prime piece on each frame? EDIT Apparently there's a 500 frame limit in photoshop gif save..." JS, p9

“All the max iteration counts for N=3 are prime numbers so far, up to a radius of 3.0 in 200 steps starting at radius 1.0 of course. I doubt the pattern will hold perfectly, but it's very interesting anyway. I'm astounded by it, actually. I'll test a longer run and look for a rule.
Primes show up about X/log(X-1) times in the first X integers.
I'll conjecture that the number of primes will be more common in the set of "max iteration" counts for N=3. I'll guess more than X/log(cuberoot(X)) I'll report back.” JS, p9

“I don't mind saying this was a lot of work and CPU time! I hope some of you watch these. 😃
I'd recommend N=10 if you want to try just one.” JS, p9

N=1 highres N=2 highres N=3 highres
N=4 highres N=5 highres N=6 highres
N=7 highres N=8 highres N=9 highres
N=10 highres N=11 highres N=12 highres
N=13 highres N=14 highres N=15 highres
N=16 highres N=17 highres N=18 highres
N=19 highres N=20 highres N=21 highres
N=22 highres N=23 highres N=24 highres

“Here's an N=3 run of 2000 frames, counting primes. Seems like they're 2.7 times more likely than usual. I think I'm a human who saw the pattern 3, 5, 7, 11, 13 and got carried away. But I've colored the prime pieces blue in the movie. Can you find a pattern that would always give us a prime piece on each frame? EDIT Apparently there's a 500 frame limit in photoshop gif save...” JS, p9



primes

Carl Hoff wrote: " Can you render this with the two full circles showing as well? The higher the resolution the better.

"The poster rendering finished. The final image is here:
http://www.brandonenright.net/~bmenrigh ... r_size.png
I rendered it at 600 DPI for a poster 24.5 x 59 inches. The image is 519 megapixels. This is way too big for most image libraries to handle so Firefox and Chrome and most things on Windows will refuse to process the image. If you want to look at it you'll need to download the file and open it in something like GIMP or Photoshop.
I did some color adjustment, added black margins, and converted it to CMYK. I've ordered a test print. I'll report back on how the print comes out in a few weeks when I get it." BE, p9

"I've been invited to present at the Gardner Celebration of Mind event at MSRI in Berkeley this fall. Given the new and ongoing results here I expect I'll talk about this, of course giving you guys credit and showing your movies and images, if you're OK with that. I thivnk I will want to spruce up my program some before then as well. In principle I should be able to generate resolution-independent polygons, which could then be rendered into movies pretty efficiently. I can also optimize quite a bit on my current code, when restricting to the single-generator case." BH, p9

“I consider all the (original) code, images, and other (original) information I've posted in this thread to be public domain so share away. 😀 I'd love to attend. I found the event page but no registration information. Do you know if it's open to the public?” BE, p9

“Nice!!! I even can see a preview on my cell phone which is all I have for the next few days. But I'm just seeing the football shaped intersection of the 2 circles. Above we were talking about a display of both full circles. Nothing new should need to be rendered just bits copied around from the intersection. But now I'm assuming this is rendered at such a high resolution the 2 full circle image would be crazy big. So maybe this is actually the better way to go. I was just thinking the 2 circles would better fill a rectangular picture frame then this football intersection shape. But now I don't want to give up the resolution.” CH, p9

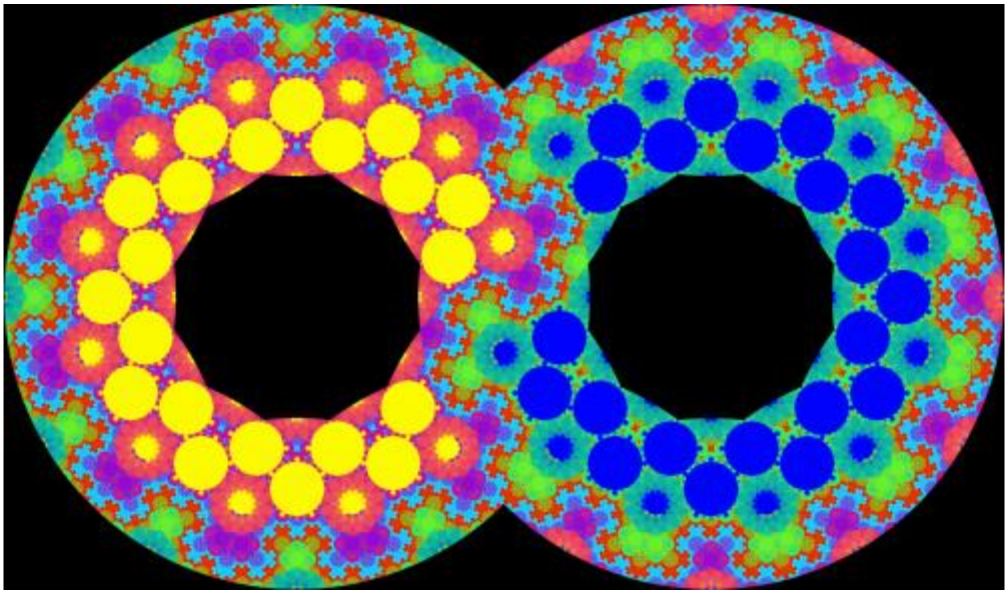
“Yes, but you do have to register, and it fills quickly. I'll let you know when I hear registration's opened. The speakers are great -- I'm honored to have been asked this time. Last year we had Don Knuth, Cliff Stoll, and a couple others.” BH, p9

“Bob, I'm interested in attending as well. Let me know when it's open if you can.” JS, p9

“I don't put art on my walls, but I'd definitely love a print when it's available. I've got a question about the anti-aliasing. In most of the image, you make a smooth transition between two different colored regions. This doesn't occur at the fractal boundary. Is this intentional or an artifact of how difficult it is to compute their order there? There's also a few places where it's not quite what I expected. Near (6510, 23338) it is brighter than either side. It reminds me of **this**, but that usually makes blending darker.” Landon Kryger, p9

“Here's my attempt to recreate the full circle with a lower resolution copy of the wedge. Kind of eyeballed it, but it seems to fit with **Bob's images**.

There may be some subtleties to this that I'm missing so I wouldn't bet the farm on its accuracy." Landon Kryger, p9



"The print came. It's pretty good quality but not quite as good as I'd hoped.

The print is way higher resolution than my photo so you can't tell what the actual print quality is. When examined up close there is a bit of blurring of the sharp lines and the fractal boundary between the orange and blue isn't as sharp as I'd like. I'm not sure if I'm just spoiled and expecting too much from a print or if higher print quality is possible. I'm going to reach out to some other poster printing services and see if they can print at a higher quality.” BE, p9

BHearn wrote: “Very pretty! If you're coming to my BBQ tomorrow, please bring it. ”

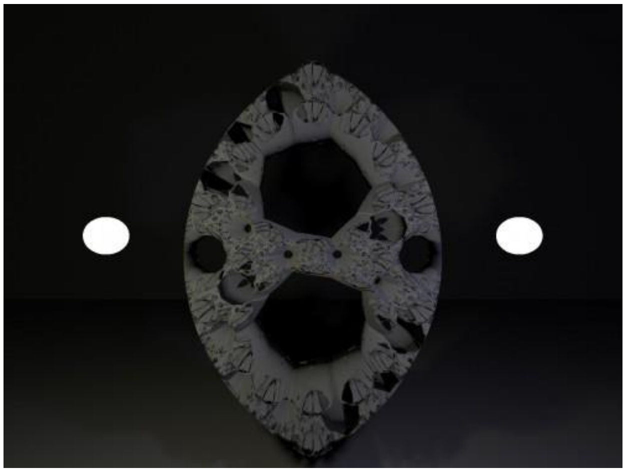
“Indeed, I will be there! I was very happy that it came today, I didn't think it was going to make it in time.” BE, p9
“If I wasn't on the other side of the country, this is starting to sound like a BBQ I'd really like to attend myself.”   CH, p9

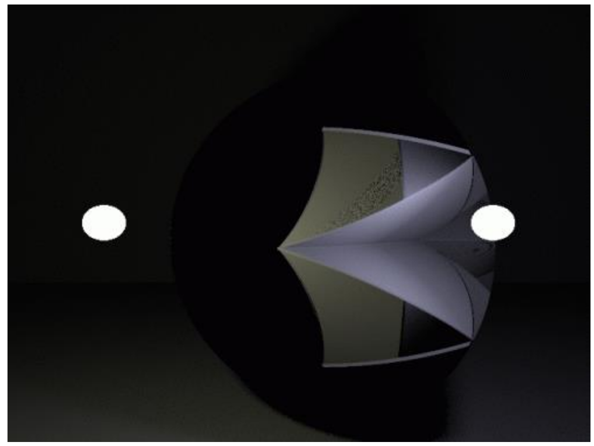
“Can't believe I had to miss the barbecue due to work. I'm making slow progress in 3d gizmo spheres and wanted to post a quick N=5 critical radius video going from 10 iterations to a few thousand.
The max iteration count forms the iso-surface. I'm sure there are some obscure bugs in here, but the main math is correct. I confirmed by generating some of our favorite 2d patterns already.” JS, p9

“At first I wasn't sure what I was looking at but I think I've figured it out. Lemme know if I have this right: You set two spheres to overlap at their deepest point by the critical radius. Because they're spheres this covers all overlaps between zero and the critical radius. Then you "erode" the volume by culling material that has an order less than some threshold. As you increase the threshold more and more of the intersection erodes and eventually all that's left is the material with a very high order?
It sure looks cool! It would be pretty amazing if you could work out the math for a colored-glass like material where the color is determined by the order. Then the intersection could be rotated with a light behind it. This would probably be way too CPU intensive though!” BE, p9

“That print looks awesome! (BE’s poster) Brandon, you're right. It's two overlapping spheres going through the same gizmo sequence A'B. I'm including a couple of images showing the arrangement. First, the two spheres, and then the intersection region where raymarching the gizmo

begins.
I'm sure some interesting arrangements are possible in 3d but I'm still up to my neck in basics right now.” JS, p9 Click the image to play.





N=5, iterations max 200

“This is a Gizmo 3D render from a 3/4 view. Still N=5.” JS, p9

“I was at MSRI from around 11:30 to around 1:30. The celebration of mind event was neat but not really was I was expecting. It was heavily geared towards kids and it reminded me a bit more of a vendor floor where people had books and toys out for people to walk by and play with or chat about. The lecture portion only lasted about 30 minutes and had 3 speakers each with about 10 minutes. The talks took more the form of a "show and tell" than a lecture. The best part was meeting Brady Haran of **Numberphile**, among other things, and chatting with him for some time. I'd love to get Brady to do some videos on various mathematical / theoretical aspects of twisty puzzles. I think there are probably two big hurdles do that though. First, his channel is really about numbers and almost all of the math chosen is closely related to numerical things. Second, he almost always interviews recognized experts in a field (people with PhDs) and the vast majority of us are doing this at a hobbyist level.” BE, p9

“Brady is a great film maker who I have followed for years. Apart from Numberphile, he has many other YouTube channels which may be a better fit with twisties. Also, you don't have to be a PhD to be an expert in your field, I think interesting and entertaining are more important. I would also note that there are already many YT channels devoted to twisty puzzles although not many explore the mathematical aspects of them." Gus, p9

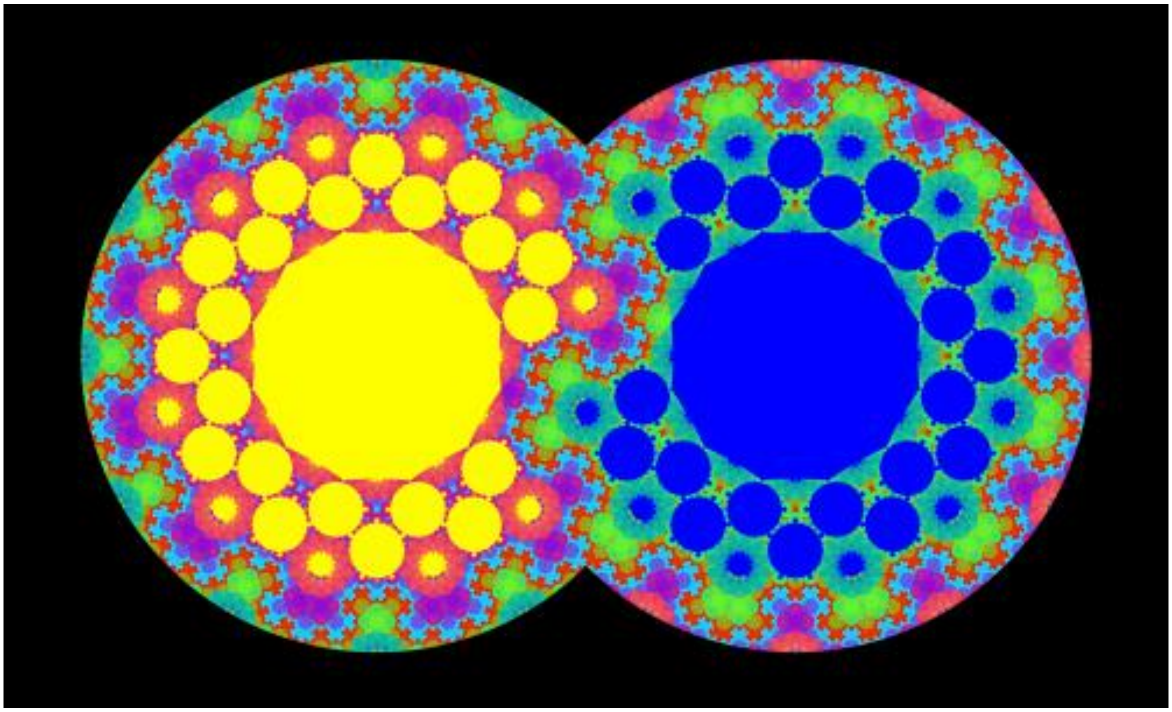
Page 10 Monday, November 2, 2015 thru Thursday, June 2, 2016

"I sat with him at dinner last night (Elwyn Berlekamp's 75th birthday celebration), and chatted quite a bit. But I didn't try to hit him up for doing a video. He's here (MSRI) at the combinatorial game theory workshop yesterday and today as well; maybe I'll get a chance to chat some more. Not sure if I count as a "recognized expert" in twisty puzzles, but at least I do have a Ph.D. Eitan spoke with him at the CoM event here last year, and seemed to think there was some chance he'd follow up to do a twisty segment, but it hasn't happened yet." BH, p10

"…Who's in the expert minority in this case?" Jared, p10

"I had Bram, Bob, Jaap, and Oskar in mind. There is no "Twisty Puzzles" degree but there are related math and engineering fields and I think those folks are pretty widely recognized as experts in related fields. Now that you ask though, I think Tom Rokicki and Herbert Kociemba are probably also recognized as experts. I think issues of "expertness" could be side-stepped with peer review." BE, p10

"I think I have a proof of jumbling in the N=5 case at the critical radius we've already worked out **here**. I'm still working out all the details to make sure it really is as simple as it seems to be.
Until I've worked out the details (hopefully by the end of the day!) I thought I'd share some other progress. At the urging of Carl and GuiltyBystander, I've extended my renderer to be able to draw the full disks in addition to just the intersection.



The version GuiltyBystander made earlier by replicating parts of the wedge is a nearly perfect approximation. I'm working on a ultra high resolution copy suitable for printing.
If anyone wants the raw data file for the image before it goes through my color algorithm, lemme know. There is a lot of detail hidden around the fractal boundary that my color algorithm does a poor job of showing." BE. P10

Brandon Enright wrote: "All of this can be seen through a careful examination of the construction and watching the movement step-by-step. (from a detailed proof of Rc by BE, not reprinted here, DE)"

CH wrote: "Can be seen" clearly doesn't mean can be *easily* seen as I think I fried a few brain cells getting me there, but I believe I see it now."

"Yeah it's easier to imagine the whole line segment being moved by a rotation instead of watching individual points. My animated GIF moves too fast so you have to watch the thing over and over.

If I weren't so lazy I would have re-made several examples showing just portions that illustrate the behavior. The whole post took me about 9 hours yesterday though and I ran out of energy for doing any more work. I'm watching animals for a coworker and the constant stream of small interruptions from them slowed me down really badly. Deep thinking is like deep sleep, tiny interruptions will prevent you from reaching it.” BE, p10

CH wrote: “I was hoping for some insight as to why this specific R was the onset of jumbling but if that is here I'm not seeing that part of it yet.”

“Well it's a proof by example. There is an R for which there is an infinite set of points that fall on the lines shown in Bob's construction.
If I showed you the R and the lines you can verify, they work. If I ask you to find the R and the infinite set without any hints that seems to be much harder. We found it by just increasing R until "magic" happened and then looked around until we saw what the magic was.” BE

CH wrote: “To me this seems to prove this given value of R jumbles but doesn't say anything about smaller or larger R.”

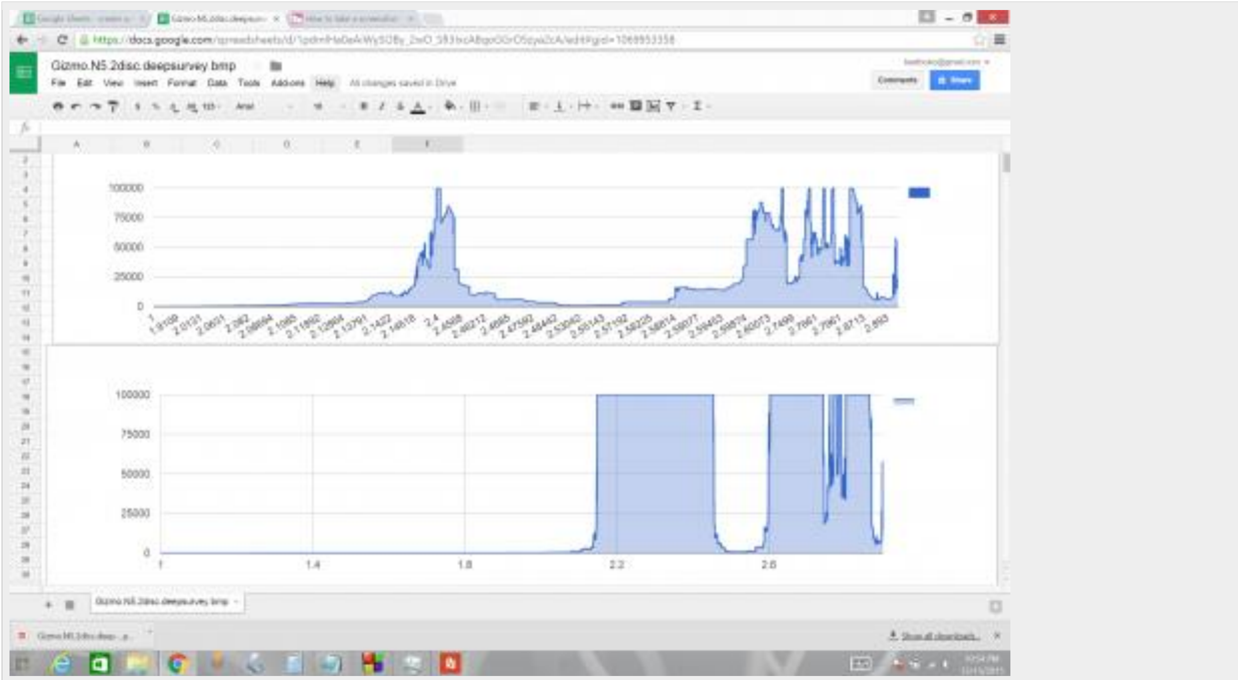
“Yes exactly. We know with some confidence through empirical search that there doesn't seem to be an infinite set as a solution for smaller R.
Now imagine the exact same line construction but R shrunk slightly. There would be short sections of the line segments that protrude beyond the radius of the circles out of reach. A point like the origin would follow the exact same path right up until it lands on one of the edges that protrudes beyond one of the circles. At this point it would "miss" one of the moves breaking the pattern that kept it always advancing by an irrational amount. I think at this point by skipping a move, all subsequent moves would actually subtract the same irrational amount and it would go all the way back to where it started. This is probably a bit of a simplification, but I think it accurately describes the phenomena.” BE

CH wrote: “I believe the consensus is that larger R jumbles and that smaller R does not. Is it possible to prove that building on this proof? We're

pretty sure smaller R doesn't by way of computer search.
For larger R it seems the single infinite set splits into two infinite sets. At some point my bet is that with a large enough R there won't be a simple line-like structure for the infinite set but we haven't explored that space much.
Right now the idea that R >= crit jumble is just a hypothesis. All my above analysis shows is that R = crit there is an infinite set of points which is sufficient to prove jumbling for just one R value." BE p10

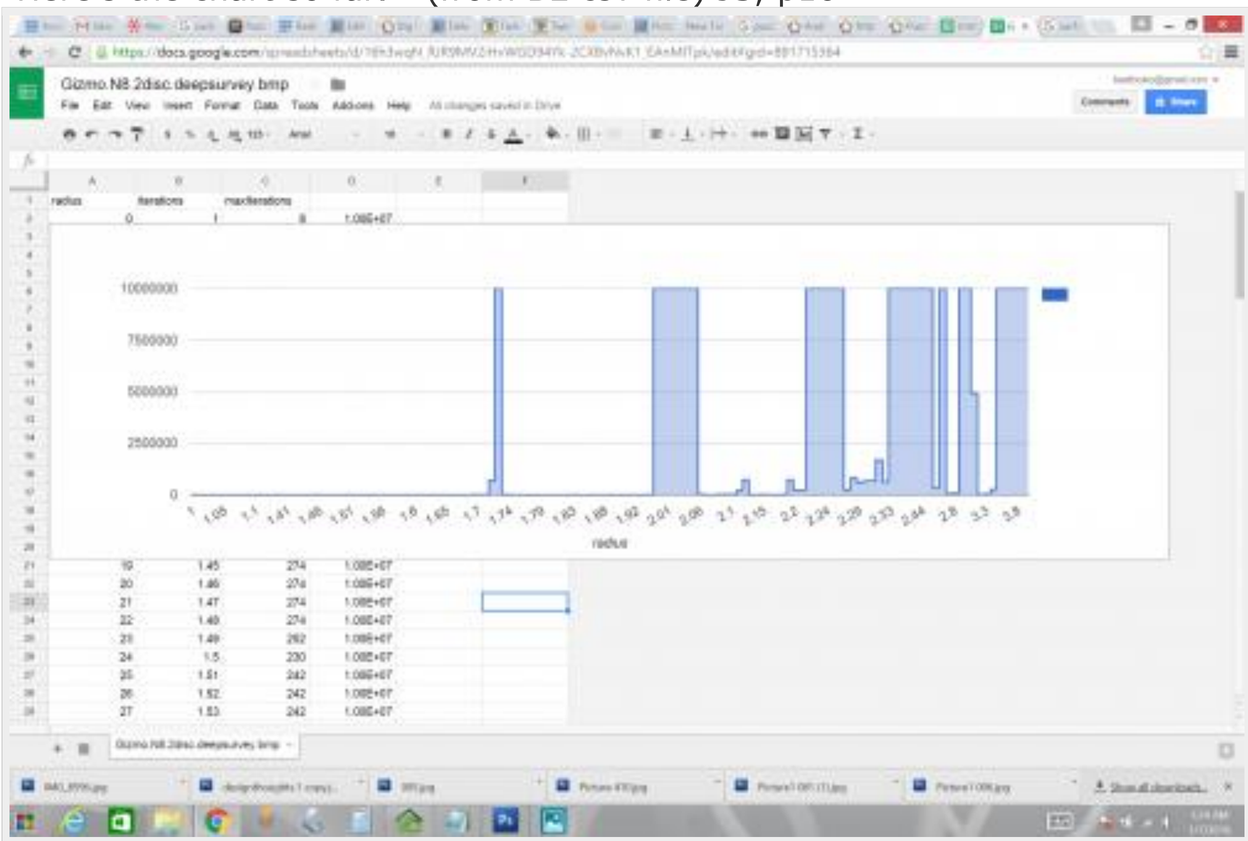
"Brandon, I'm very excited to really read this thoroughly and understand what you've done! I haven't yet, but I'm hoping to soon. Rather than a nice meaty theoretical contribution, I'm still working on surveying the space. I repaired a half dozen broken laptops over Thanksgiving. A couple of days before Christmas eve, I set them up in my attic, and launched some big batch jobs to survey N= 5, 7, 8, 9, and 10 in a new way.

The new code will evaluate a range of radius in steps of a certain size (0.1) Then, for any two steps where the max iterations changed, it will evaluate that range in 1/10th of the original step size (0.01). This is repeated for 6 decimal places.
The data is output as csv and can be graphed to show how iterations change as radius changes.
They're still chugging along! When they finish, I should do [A B] next." JS p10



“This is a very good idea. I combined it with Bob's idea of only sampling points along the perimeter of one of the disks (parameterized by theta) and now my code can find the maximum order of a puzzle MUCH faster than sampling tons of points in the overlapping wedge.
I implemented it as a recursive divide-and-conquer algorithm that keeps splitting segments who's endpoints don't match in half and then descending into the smaller segments. I have some knobs to control the minimum and maximum gaps (your .000001 and .1). Overall it seems to work very well and it's way faster than sampling the point at theta at fixed steps. Too bad PARI/GP isn't threaded because this recursive algorithm is a natural fit for parallel computing” BE, p10

“Here's the chart so far.” (from BE csv file) JS, p10



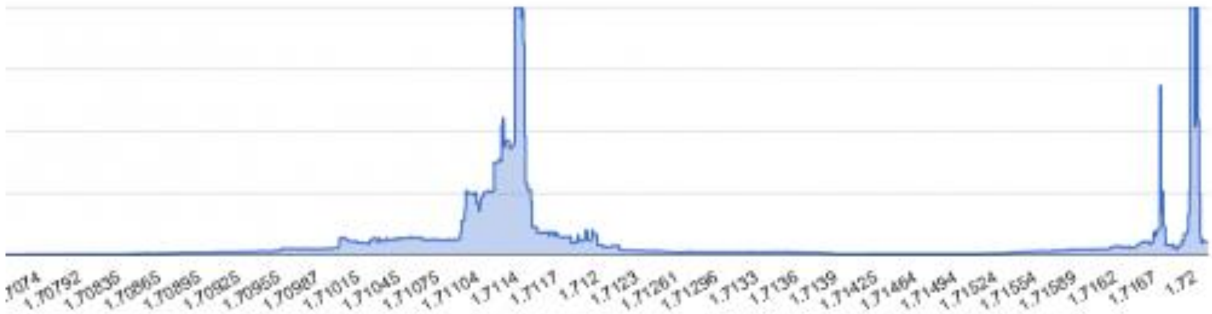
(here you can see how a specific generator goes fractal and non-fractal for various R) DE

“For N=8, [A' B] I see a very sharp spike above 10,000,000 at 1.7114
Another spike at 1.7171, but it didn't truly max out. Only hit 6,000,000 or so.
Another spike at 1.7195 - 1.7203 above 10,000,000.” JS, p10

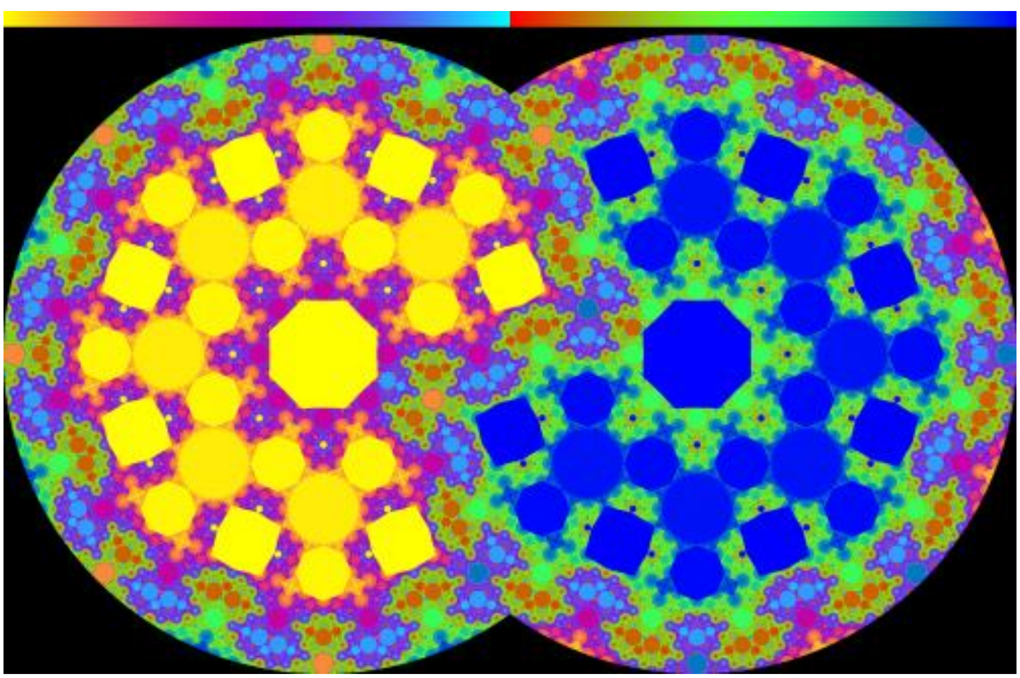
“Honestly I'm just as curious what enables the set to go finite again as R increases for some of these generator cases. Said another way... what can turn on jumbling as R decreases? (Do we still call this jumbling when we are looking only at generators? Said as I believe nothing can turn on Jumbling with decreasing R when the full puzzle is considered. Bob has an argument for this that I'm not certain I fully understand.)" CH, p10

"Jumbling with a generator is very sensitive to everything happening *just right*. Take the N=5 case that I worked on above where there is some line segment that points keep jumping by fixed intervals on. Imagine that if you make R larger then line segment gets longer but the amount a point move stays fixed. The amount a point moves is determined only by the composition of A' B and not the radius so this isn't a stretch to imagine. If you change R such that the line segment becomes a rational multiple of the jump amount the set would suddenly stop being infinite and would become periodic. Put another way, the ratio between the movement amount and the line length is 2 + sqrt(5) but changing R a tiny bit changes this ratio a tiny bit. Instead of it being 4.23606797... you could make it exactly 4.23606 and then points on the line would be periodic and have a finite order.
Eventually as you increase R enough the whole path the points on the line gets disrupted. Unless there is some other infinite set that isn't disrupted you'll see the jumbling go away.
It *seems* like with arbitrary moves instead of a fixed generator whenever something gets disrupted you can use extra moves to get the point back into the groove where it can continue on its infinite orbit. You don't have this sort of luxury with a fixed generator." BE, p10
(This clarifies the circle puzzle aspect of periodic orbits – they repeat if you use a repeating generator) DE

"1.7112 appears to be big (3 million) but it doesn't max out for me over 10 million (possibly using more decimal places in the fractal generation might see a fractal instead of maxing out)DE until 1.71133 and stays maxed until 1.71143, 1.71141 appears to be the center of the spike." JS, p10



"Here is N=8 at R=1.71133 using G=[A' B]:ote I've added a color legend to the top of the image. All the way at the left is yellow which is when a point is turned only by A (the left disk). All the way on the right is blue which are points only turned by B. In the middle is the sharp transition from teal to red which is the points where the ratio A/B for the point orbit goes from > 1 (teal) to < 1 (red).
That boundary that zig-zags through the center sure is neat. The highest order points are actually in tiny circles dotted nearby along the border." BE, p10



JasonSmith wrote: "Hmm. N=8 may jumble using [A B] at 3.25
Brandon, if you have a moment and are able to check this out as well, please let me know.
It seems like a narrow, hot spike, but needs confirmation. Pretty good spike at 2.798, too." JS, p10

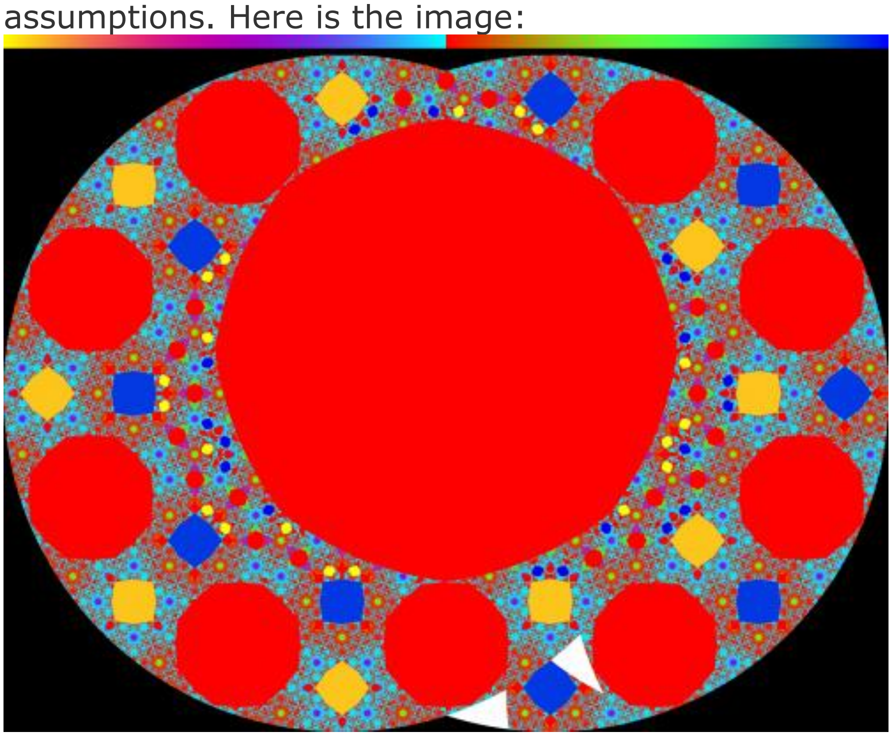
"Well there goes my [A B] never jumbles hypothesis. My code is saying:

CODE: SELECT ALL

```
? maxedgeorder_rec(8, 3.25, 1, 1, 50000000, 1000)
[17450008, 0.031942157935729632018020616993764578426]
```

Which you can read as N=8 R=3.25 G=[A B] Cutoff=50M
ThetaSteps=Segment / 1000
And the point at theta ~= 0.0319421579357 has an order of 17m. I bet if I let the border segment get subdivided more than 1000x there will be a point that goes over the 50m threshold.
Very interesting... I'm really curious what's going on here..." BE, p10

"The N=8 G=[A B] at R=3.25 kind of breaks my renderer's assumptions. Here is the image:



First, the big red circle in the center and the 9 medium sized red circles around it are all points with a finite orbit but their ratio of A to B turns is exactly 1 and my code gives that the color red. The ratio is 1 for all those points because (AB)^N = I for those points. They could just as easily be colored teal but my code happens to favor red in the case of a tie. I should modify how my renderer works so that it can detect this sort of behavior and give those points a medium gray color instead.
All the red and teal points outside of those circles have A/B ~= 1 because they have infinite (or near infinite) orbits.
I'm not sure what's going on with the two white triangles in the lower-right. White is my code telling me it can't get a sample for those points. I think what's happening is that any point that starts in one of those white triangles can never re-enter the triangle so my code can't measure their orbit length." BE, p10

Brandon Enright wrote: **"**All the red and teal points outside of those circles have A/B ~= 1 because they have infinite (or near infinite) orbits."

"Is that true for the 19 smaller red circles? My intuition is telling me they must have rather small orbits." CH

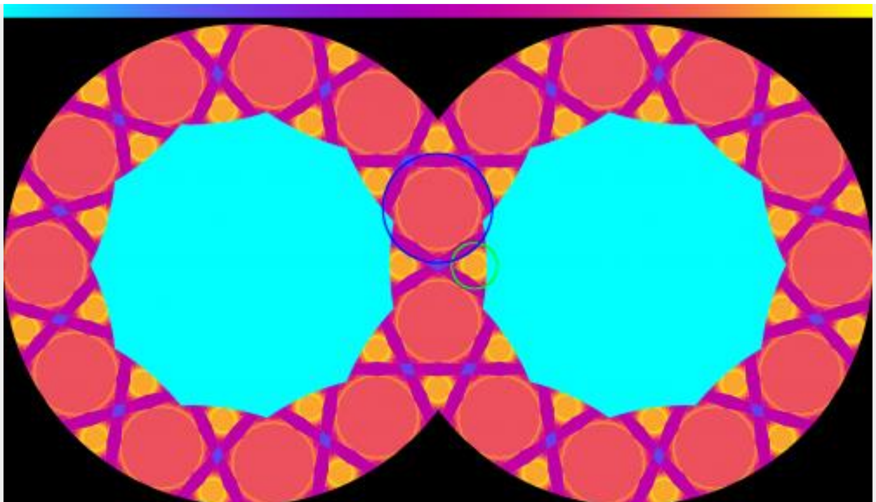
Brandon Enright wrote: **"**I'm not sure what's going on with the two white triangles in the lower-right. White is my code telling me it can't get a sample for those points. I think what's happening is that any point that starts in one of those white triangles can never re-enter the triangle so my code can't measure their orbit length."

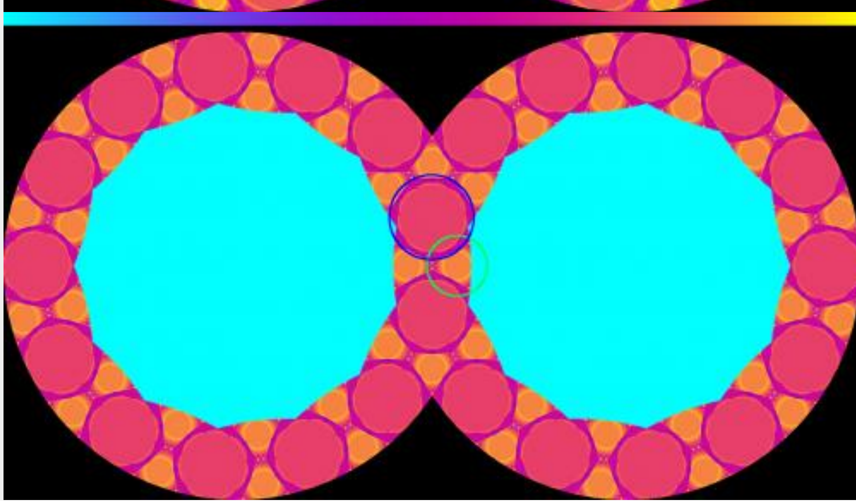
"That doesn't seem to make sense to me. The volume within those triangles must come from somewhere after [A B]. If it's coming from an area that is colored, I'd expect it to be colored as well. That leaves the possibilities that the two white areas just cycle back and forth, or they remain unchanged. I'm pretty sure its not the later but it was either of those two the orbits would be very short. Looking elsewhere on the image I can see what I think belongs in those triangles." CH, p10

"So N=8; G=[A B]; R=3.25 probably doesn't jumble but it seems really close. I increased my theta step resolution 10x and the highest order point was 32755984 at theta ~= -0.1887351543414024

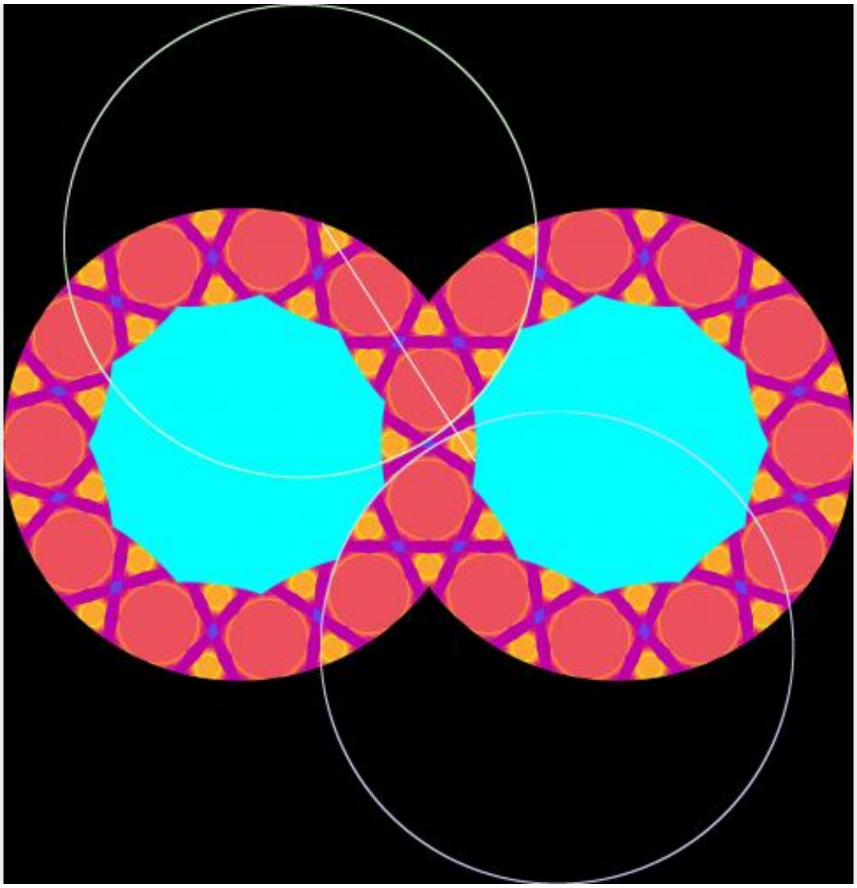
FWIW (for what it’s worth) the theta step size is (2 * acos(1 / 3.25)) / 10000 ~= 0.0002516059 radians” BE, p10

“…It seems like when N is large (>= 11) and odd, as R increases before reaching critical, you always end up with two highish order structures shown here circled for N11 and N13

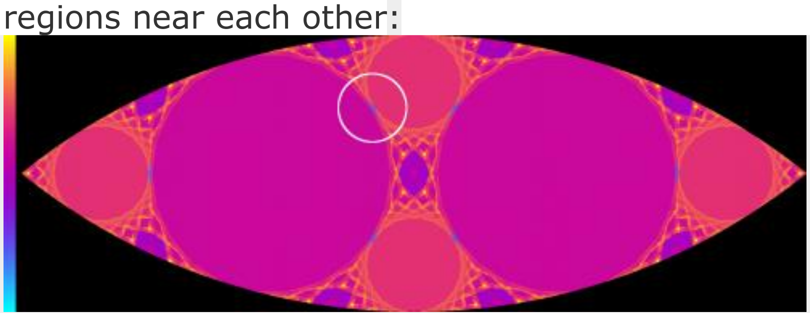




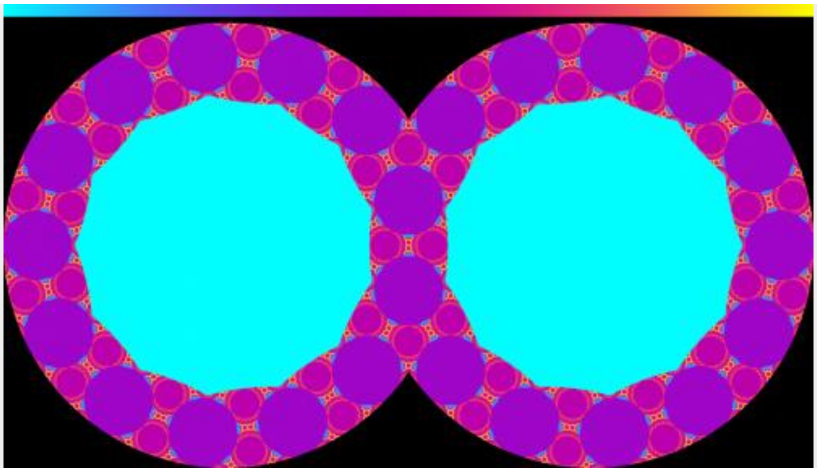
If you watch the animations closely, jumbling appears to happen when the high order structure circled in green touches the high order structure circled in blue. The boundary between those two areas is defined by these two circles and the location they first meet (as R is increased) is along the line shown:



As R continues to increase the gap gets narrower and narrower as the two regions near each other:



I have written a numerical solver to find the value for R such that the two circles shown above touch each other at this point. Here is N=13 at that value:



Jumbling doesn't seem to start **exactly** at the point these two regions meet but extremely close to it.

Here is the R value where these two regions meet for a bunch of N:

```
11 1.2884828226509780065450215305228839480
12 1.2466193564638215178581037694630870792
13 1.2132135324690648102334229491617059023
14 1.1861077449375551566056259861411378887
15 1.1638081572948309322453617163579564484
16 1.1452447106052806979037678135457828410
17 1.1296311871922219806902277917904924849
18 1.1163781200175322487971976806513331177
19 1.1050360518188911074681240618802338856
20 1.0952572878511717583260629286164456251
21 1.0867694191114482016408957821516769360
22 1.0793565913194180018721630097383757302
23 1.0728460086988045064309685929893314349
24 1.0670980542660694957539404387783078182
25 1.0619989557449688834851128735258826339
26 1.0574552726612212636783299351603289411
27 1.0533897051910301484967604879208125364
```

I'm not so sure if this is a good estimator when N is even but it seems to be a pretty good estimate (possibly a lower bound) when N is odd.” BE, p10

“I spoke recently with Brandon about some of the subtleties of the definitions we're using and the infinite processes we've seen, and we decided it would be better to discuss those ideas here.

In this thread, we've generally used the definition that a two-circle puzzle of radius R jumbles if a point on its boundary (i.e. a point of distance R to one of the centers) has an infinite image under rotations of the puzzle. However, many posts have looked at points not on the boundary, and I realized that even at the critical radius, it might be possible for a point NOT on the boundary to have an infinite image under rotations such that no point in the image is on the boundary. One could imagine that on the puzzle of critical radius, there exists a point in the interior (i.e not on the boundary) of the puzzle such that the image of the point contains points arbitrarily close to the boundary, yet contains no points on the boundary.

I think we've even seen some evidence for points like these. Consider N=5 at the critical radius. Bob **found a point on the boundary** whose **image** under one generator looks like several straight line segments. However, the image of the point must be countable, since the set of finite sequences of moves is countable. Each line segment consists of uncountably many points, so the image of this point cannot actually contain a whole line segment. This means that we can pick a point on one of these line segments whose image does not contain the original point on the boundary. I suspect that we can even pick such a point whose image contains NO points on the boundary. On the other hand, assuming **Brandon's math** is correct, such a point must have an infinite image on the line segment.

Such a point would also have some weird properties when we try to consider its behavior on the fully unbandaged puzzle. On the one hand, this point must not lie on the boundary of any part, since then its image would lie on the boundary of the puzzle. But on the other hand, such a point cannot lie in the interior of some part with well-defined area, since then the image of this point would be strictly contained in a smaller puzzle (i.e. a puzzle with radius less than the critical radius), which would imply that the image of this point is finite! So, either these points do not exist, or the notion of a "part" on the fully unbandaged puzzle is not well-defined. I suspect that such parts might be **sets of measure zero** or **non-measurable sets**.

It's also worth noting that if we can find a point in the interior that has an infinite image which gets arbitrarily close to the critical radius, this could possibly help explain the existence of this critical radius. One might expect that at any radius less than the critical radius, this image gets "cut off" before it can become infinite. Furthermore, if we can prove that an interior point of the puzzle having an infinite image implies that a boundary point of the puzzle has an infinite image, we could conclusively prove jumbling for all radii greater than the critical radius

So my question is: can points on the interior of a puzzle at the critical radius have infinite images which never touch the boundary? If so, what is the relation between the existence of these points and the existence of points on the boundary with infinite images?" Will_57, p10 **My website** | **My designs available for download** | **My Shapeways models**

"Such a point would have to exist in an area which was cut up into what we've called jumble dust in the past. As its the boundary which does the cutting, you are basically saying there are points in this area which the boundary never touches. It would seem that the distance from this point to the boundary must approach zero as more and more cuts are added. So would it be valid to say that such a point was on the boundary after an infinite number of cuts had been applied? Not certain... but if so you seem to be drawing a distinction between points which end up on the boundary after a finite number of cuts have been applied and those which require an infinite number of cuts to get there. And I'm even less certain if such a distinction is valid and/or meaningful. It is certainly an interesting question but I don't think I have the answer you are after. This is a new enough area of exploration that I'm not even certain its a valid question, and if it is I don't think anyone has answered it yet." CH, p10

Brandon Enright wrote: "I think there is an analogous but slightly easier to think about and understand scenario where something like this happens. Take the number line and imagine cutting the number line up into a bunch of line segments where the ends of each line segment must be a rational number."

"So you could cut a line segment from 1/4 to 1/2 or from 2 to 3 or 1/3 to 1/5, etc. If you pick a rational point on the number line then that point can fall on the bounds of a line segment. If you pick an irrational point like sqrt(2) then no matter how much you cut up the line segments, you can never have your sqrt(2) point fall on boundary of a line segment.

So even after a (countably) infinite amount of cutting where the length of the line segments approaches zero, there are still points that must be on the interior of line segments even though those line segments are of length zero in the limit.

Ahhh... so this question seems answerable." CH

CH wrote: "So would it be valid to say that such a point was on the boundary after an infinite number of cuts had been applied? That answer appears to be "no". CH

Brandon Enright wrote: "I think what Will has done is demonstrated the existence of points on the line that must always lie in the interior of pieces."

"Ok... going back and re-reading Will's post in light of yours I tend to agree. Though I'm not certain I can wrap my mind around the "interior" of a zero-volume part. Though I have the same problem with the interior of zero length line segment too.

Going back to your sqrt(2) example... yes its irrational. You cannot express sqrt(2) as A/B with A and B being finite integers. But if you are looking at countably infinite cutting why do we need to limit ourselves to finite integers. Couldn't we say:

Pi = 314159.... / 100000.... where both the numerator and denominator have the same number of digits and that number of digits is countably infinite.

I agree that Pi never falls on the end of a line segment for any finite amount of cutting but I'm just not sure it's well defined for when the amount of cutting actually is infinite. Just as I have no idea what to call a number with an infinite number of digits. I hear people talk about finite integers all the time. Is there such a thing as an infinite integer? Even if there is its likely totally meaningless to talk about dividing them.

Looking at your example from the other direction... any rational number will end up on the boundary of a line segment in a finite number of steps. So, what kind of points are at the boundaries after an infinite number of steps? If you cover all the rationals before then doesn't that just leave the irrationals?

I think I've broken just about every mathematical rule I know so consider this me playing devils advocate as I'm not sure I really buy my own arguments here.
Infinity sure is a tough concept to wrap your mind around," CH p10

CH wrote: "I agree that Pi never falls on the end of a line segment for any finite amount of cutting but I'm just not sure its well defined for when the amount of cutting actually is infinite."

"I don't know the answer to this however lets assume for a moment that after an infinite amount of cutting you can approach one irrational point like Pi and reach it exactly in the limit. The problem is that there are an unaccountably infinite number of other points in the line segment that are also irrational. Even if you could reach one with a countably infinite number of cuts, you can't reach them all at the same time with a countably infinite number of cuts." BE

CH wrote: "Infinity sure is a tough concept to wrap your mind around"

"Yeah and worse, we're talking about the cardinality of infinite sets where the **cardinality of the continuum** is strictly greater than "infinity"." BE, p10

"I've answered one of my questions. Once again, recall **this image** of N=5 at the critical radius. We've already proved that every point on one of these line segments has an infinite image. Here I'll prove that there exists a point on the middle line segment whose image never touches the boundary after any finite number of rotations. (In fact, **almost all** points on this line segment have this property, but I don't think this is of much importance). Consider a line segment between any two points on the boundary of the puzzle. Note that after a single rotation, the image of this line segment consists of at most three line segments, all of which have endpoints on the boundary of the puzzle. The key observation here is that after any finite number of rotations, the image of a line segment consists of finitely many li

ne segments, which intersect the boundary of the puzzle in finitely many

locations. Let S(*n*) be the set of points on the middle line segment such that some sequence of *n* rotations moves this point to the boundary of the puzzle (e.g. S(0) consists of just the endpoints of the line segment). Formally, we just showed that for every integer *n*, S(*n*) is finite. Since a point on the line segment has an image on the boundary if and only if it is in S(*n*) for some *n*, the set of points on the line segment that have an image on the boundary is the union over all integers *n* of S(*n*). This is a countable union of finite sets, which is necessarily countable. Thus, the set of points on the line segment with an image on the boundary is countable. The set of points on the line segment is uncountable, so there exist points on the line segment with infinite images that lie strictly on the interior of the puzzle.
Note that while the images of these points never reach the boundary in any finite number of rotations, they are closely related to the points that do; they can be arbitrarily approximated by such points. Additionally, these points have images that get arbitrarily close to the boundary. It would be interesting to see if this behavior is true in general.
Something else that has been bothering me: most seem convinced that if a puzzle jumbles in the traditional sense (i.e. every finite unbandaging can be unbandaged further), then there exists a point on the boundary with an infinite image. Yet, one could imagine a puzzle where every point on the boundary has a finite image, but for every integer *n*, there exists a point on the boundary whose image contains at least *n* points. How can we conclusively rule out this possibility? I hope I'm not missing something obvious here..." Will_57, p10

"Forgive me, I lack a lot of knowledge of this area of mathematics. What you've described here seems either contradictory or so subtly similar to having a point of infinite order that the difference is meaningless. Is there a name for the property you describe so I can read up more on why the distinction matters?

I can't imagine a situation where all points on the boundary have an image that's well-defined and finite yet you can find points with arbitrarily large images. f(x) = 1 / x has the property that you can find values of arbitrary size yet f(0) is not infinity, it's undefined. I'm not particularly imaginative

but I can't think of any situation where everything is defined and finite yet unbounded. I'm inclined to argue something along the lines of:
All points are finite. Therefor there is a largest point. That point is finite. Therefor the puzzle doesn't jumble. I don't get how there can be an unbounded-yet-finite situation that defeats this argument." BE, p10

Brandon Enright wrote: "Is there a name for the property you describe so I can read up more on why the distinction matters?"

"The function which maps points on the circle to the (integer) number of points in its image would have to be **unbounded**. On the other hand, if the image of some single point was infinite, such a function would not be definable because "infinity" is not an integer." Will_57

Brandon Enright wrote: "I can't imagine a situation where all points on the boundary have an image that's well-defined and finite yet you can find points with arbitrarily large images. f(x) = 1 / x has the property that you can find values of arbitrary size yet f(0) is not infinity, it's undefined. I'm not particularly imaginative but I can't think of any situation where everything is defined and finite yet unbounded."

"I think it's intuitively easy to understand that an unbounded function cannot be very well-behaved. It turns out that every continuous function defined on a closed and bounded subset of Euclidean space must be bounded, and most functions which we think of as being "nice" satisfy these properties. So, to make your example work, we could either define it on an open interval (e.g. define the function only on (0,1)), or make the function discontinuous (e.g. define f(0) = 0)." Will_57, p10

Brandon Enright wrote: "I'm inclined to argue something along the lines of:
All points are finite. Therefor there is a largest point."

"This is not true. Every integer is finite, but there is no largest integer." Will_57, p10

"Yeah I walked right into that one. So is there any difference between a jumbling puzzle with a point on the border with an infinite image and a puzzle where there are no infinite-image points but there is no bound on the image size?” BE, p10

“It's hard to say because I have doubts that such a puzzle could even exist, particularly if it's a Gizmo Gears-type puzzle. One property we discovered when a single point had an infinite image was that the jumble dust can contain points that don't belong to a "part" in a traditional sense, unless you're willing to accept that a part might consist of a single point. It could be the case that if every point on the boundary has finite image and these images are unbounded in size, then you get more well-behaved jumble dust. This would be to say that the unbandaged puzzle consists of a countably infinite number of parts, wherein each part has a well-defined boundary, interior, and (positive) area.

I'm just throwing out ideas, but I think these are the type of properties that would be interesting to investigate if we find an instance of such a puzzle. Otherwise, things like these would be good places to look for evidence that puzzles like this cannot exist.” Will_57, p10

“I had a similar line of thought this evening while thinking about your finite-but-unbounded idea.

There are certain properties of these puzzles that by construction I think we'll all agree these puzzles should have:

1) There are no gaps. Cutting doesn't remove area, just splits it into regions

- 2) Parts are continuous (no holes) convex regions
- 3) All points on the border of parts have points on the border of the disks in their image
- 4) Parts have well-defined positive area (but potentially are arbitrarily small)
- 5) All the points in a part have the same order for their image

If we let any points have infinite order I think we run into trouble with one or more of these assumptions.

But if we instead say point in the borders have finite but unbounded images I think we sidestep all of the problems.

One thing I think we have to be careful of, and I'm not sure the right answer to, is how cuts actually work. Say we cut the number line between [0, 2] at the point 1. That makes two intervals but are they [0, 1) and (1,2]? Perhaps the cut points don't belong to any piece? I wouldn't call this a gap but it might be that the cut point itself is undefined as to what part it belongs to or what order it has. In the same way, speaking of the points on the N=5; R=crit lines as "infinite order" might be incorrect. They might be undefined instead. Then we aren't talking about points on the border having an infinite image but that they're undefined and points approaching them have finite but arbitrarily large images." BE, p10

Brandon Enright wrote: "One thing I think we have to be careful of, and I'm not sure the right answer to, is how cuts actually work. Say we cut the number line between [0, 2] at the point 1. That makes two intervals but are they [0, 1) and (1,2]? Perhaps the cut points don't belong to any piece?"

"It is times like this that my math side and my physics side are at odds with each other. My math side is finding this discussion fascinating. I enjoy thinking about cardinalities and I wish I had something I could contribute here.
My physics side looks at this and thinks... if we are arguing over the difference between unbounded and infinite we are splitting atoms here... literally. Our Gizmo Gears puzzle contains a finite number of atoms so it will never have more then a finite number of pieces. On the line between [0, 2] if we cut it at point 1, odds are there isn't an atom at that exact location anyways so it will either end up being [0, 1] + (1,2] or [0, 1) + [1,2] and that assumes the cutting tool doesn't carry away a few atoms itself. Even if there is an atom at point 1 I can assure you that your cutting tool will still push it to one side or the other, that is if it doesn't carry it away altogether. Don't get me wrong... the math is probably more fun to think about but the disconnect between math and reality sometimes strikes me as so big that I sometimes wonder how these mathematical models are advanced as they are. Something about that strikes me as similar to trying to design a TV which has a resolution that is higher then reality. My degree is in Physics, I'm sure someone with a more extensive math background would have a totally different perspective. And honestly... there probably isn't much that I would enjoy more then having that conversation with them over that difference in perspective." CH, p10

Brandon Enright wrote: **"**One thing I think we have to be careful of, and I'm not sure the right answer to, is how cuts actually work."

"I see at least two possible conventions:

1. Parts include their boundary. This has the advantage that every part is closed, even after infinitely many cuts, since then each part can be expressed as a countable intersection of closed sets, which is **necessarily closed**. The problem with this convention is that the boundary of a part will inevitably belong to more than one part.

- 2. Parts don't include their boundary; the boundary is either a separate entity or removed from the puzzle entirely. This is to say that parts are defined by their interior. While this solves the problem of points belonging to multiple parts, this creates the problem that a part might have empty interior.

I tend to believe that the first convention works better, because we've already seen evidence for parts that consist of a single point, and I don't like the idea of an empty part.
CH wrote: "My degree is in Physics, I'm sure someone with a more extensive math background would have a totally different perspective.

Currently pursuing a math degree here, can confirm 😜." Will_57, p10

"Warning: long post and math ahead.
I have a few more interesting results. On the question of how to deal with the boundary, I've accepted that no matter what we do, the boundary is going to behave poorly. This is because the function that rotates points in a fixed circle can never be continuous on the boundary of the circle, no matter how we define its behavior there.

On the other hand, the function is continuous at every point of the interior of the circle. This allows for a very useful property: given a point A whose image always lies on the interior, after any finite sequence of moves G, there is some open neighborhood around A that also moves with A as we perform the rotations in G. Per a suggestion by a friend, we could call such a neighborhood a “preservable neighborhood” of A with respect to G. Intuitively, this corresponds to the notion that if a point has finite image and its image never touches the boundary, then that point is on the interior of some well-defined part.

One property I have noticed with regards to infinite images is that they always seem to contain no **isolated points**. This is to say that whenever a point has infinite image, it seems to be the case that the image is **dense-in-itself** (i.e. any point in the image can be approximated arbitrarily well by other points in the image). Here I’ll partially explain why this is true: let A be a point whose image is infinite and contained on the interior. Then either every point in the image of A is isolated, or no point in the image of A is isolated. We will prove this by showing that if one point in the image of A is not isolated, then every point in the image is not isolated. Without loss of generality, let A fail to be isolated in its own infinite image. For any point B in the image of A, let G be a finite sequence of rotations that moves A to B. A has a preservable neighborhood with respect to G. Since A is not isolated in its image, the image of A must have points arbitrarily close to A in this preservable neighborhood. Applying G to any of these points produces points in the image of A that are arbitrarily close to B. This shows that B is not isolated in the image of A, as desired.

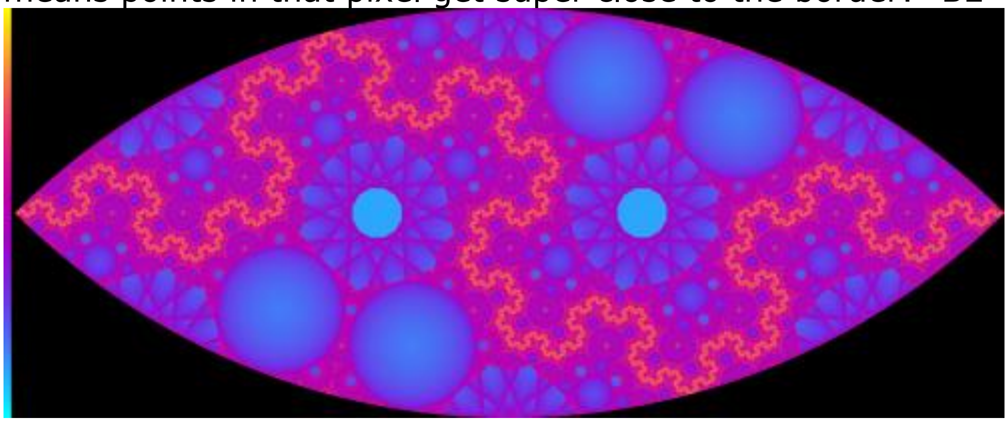
Another fact (which is clearly true at the critical radius but harder to imagine in general): once again, let A be a point whose image is infinite and contained on the interior. Then the image of A contains points arbitrarily close to the boundary. Suppose for a contradiction that this is not the case. Then there is some open neighborhood of A that is a preservable neighborhood of A with respect to every finite sequence of moves. In particular, if A never gets closer than epsilon to the boundary, then the open circle of radius epsilon around A will suffice. Now, do the same for every point on the plane that belongs to a preservable neighborhood of its own. Formally, for every point X in the plane that has a preservable neighborhood with respect to every finite sequence of rotations, let C(X) be an open circle that is contained in this neighborhood. The radius of choice might depend on the point, and in general these radii can be arbitrarily small. Define an equivalence relation “#” on points of the plane such that X#Y if there exists a finite sequence of points Z_0,...,Z_k with Z_0=X and Z_k=Y such that C(Z_i) intersects C(Z_i-1) for all integers i with 1<=i<=k. Intuitively, X#Y is a sufficient condition for X and Y to belong to preservable neighborhoods of each other. Now, let C be the union of C(X) over all X such that A#X holds. Note that C is open and connected, since it is a union of open sets, and if it were disconnected, we could find two points that do not satisfy the equivalence relation. Informally, you can think of C as being the largest preservable neighborhood of A that has these properties, but we will not prove that this is the case. Also, recall that the preservable neighborhood around A contains the open circle of radius epsilon around A, so this circle must also be contained in C. Since the image of A is infinite, there exist distinct points A_1, A_2 in the image of A with d(A_1,A_2)<epsilon (this is because every infinite sequence on a compact set has a convergent subsequence, which must be Cauchy). If G_1 is a finite sequence of rotations that moves A to A_1, let C_1 be the image of C under G_1 (likewise for G_2, C_2). Thus we have shown that C_1 intersects C_2. By definition, if these two preservable neighborhoods intersect, then they must actually be equal, i.e. C_1=C_2. On the other hand, C_2 can be obtained from C_1 by a nonzero translation and a possibly zero rotation (i.e. translate from A_1 to A_2, then rotate around A_2 by some multiple of 2*pi/N). Performing these operations on at least one point in C_1 yields a point outside C_1, since C_1 is open. This is a contradiction because we have shown both C_1=C_2 and C_1!=C_2.

Once again I apologize for the wall of text, though this time I went over these proofs with a friend (who did not ask to be named here), and this was the best we could come up with. Notably, both proofs make use of the fact that the images are always contained on the interior, though I suspect there are ways to extend these proofs to points on the boundary. One way might be by showing that "almost all" points in the infinite image don't intersect the boundary. Do note that this last proof suggests something interesting: instead of counting how many points are in the image of a single point, we could look at how close the image of a point gets to the boundary of a puzzle. Would anyone be interested in modifying their simulator to visualize this? Of course, this might quickly exceed the precision of any reasonable simulation, so it would not surprise me if these images turn out to be not useful.

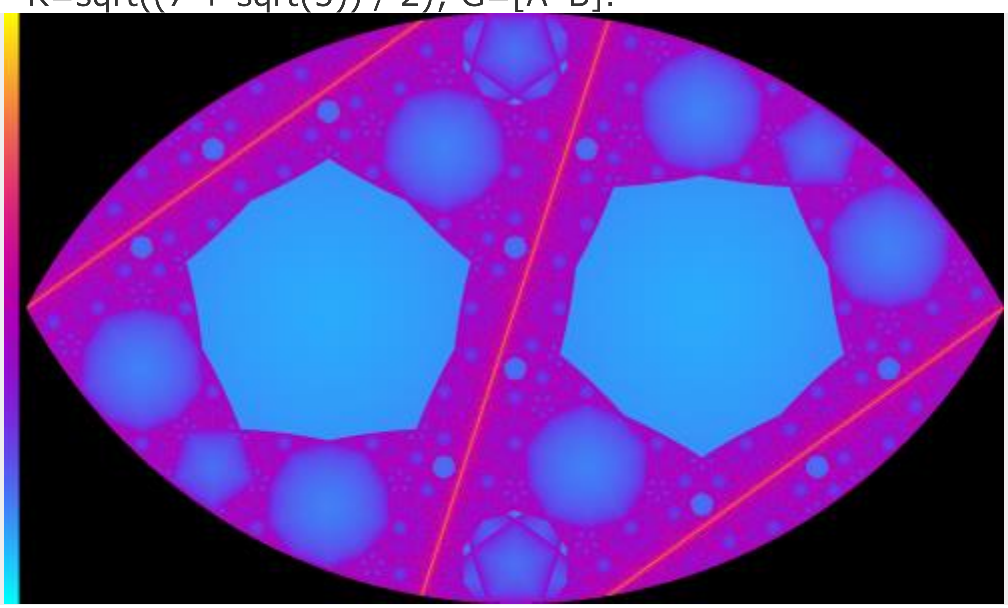
EDIT: I just realized a small mistake in the second proof. It is quite possible that the preservable neighborhood is equal to some rotation and translation of itself if the neighborhood has rotational symmetry. A good example would be a point close to the center of a puzzle of radius less than 2. Nevertheless, this is easily fixed. Instead of choosing two points in the image that are within epsilon of each other, we choose N+1 points that are all within epsilon of each other. Then, we can be sure that around at least two of these points, the image of the preserve-able neighborhood is rotated to the same orientation, which is to say that the images are nonzero translations of one another. Now, it is certainly the case that a nonzero translation of a bounded open set is not equal to itself.” Will_57 p10

Will_57 wrote: “Would anyone be interested in modifying their simulator to visualize this? Of course, this might quickly exceed the precision of any reasonable simulation, so it would not surprise me if these images turn out to be not useful.”

“This isn't as simple as it seems. For the pixels along the border, I'm finding that random sampling sometimes by "luck" comes extremely close to the border and causes that pixel to get a very high closeness score. This is causing an inefficient allocation of color in my images. I'm still trying to figure out how I can prevent lucky samples from affecting my color ranges. In the meantime, here is N=12; R=sqrt(2); G=[A' B] colored where yellow means points in that pixel get super close to the border:” BE



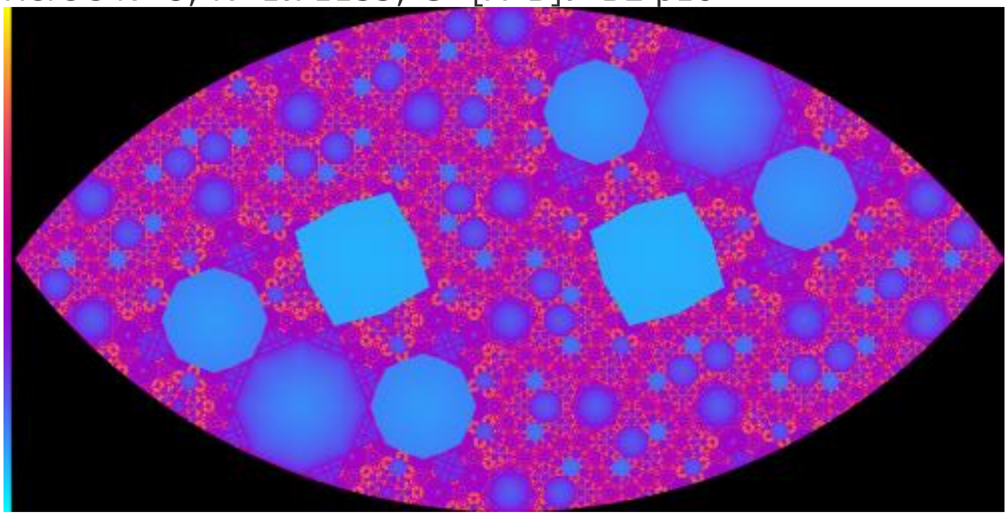
“Using the border-distance coloring here in N=5;
R=sqrt((7 + sqrt(5)) / 2); G=[A' B]:



Once again, there are some troublesome border pixels that got a lucky sample. I'm still not sure how to work around the "problem

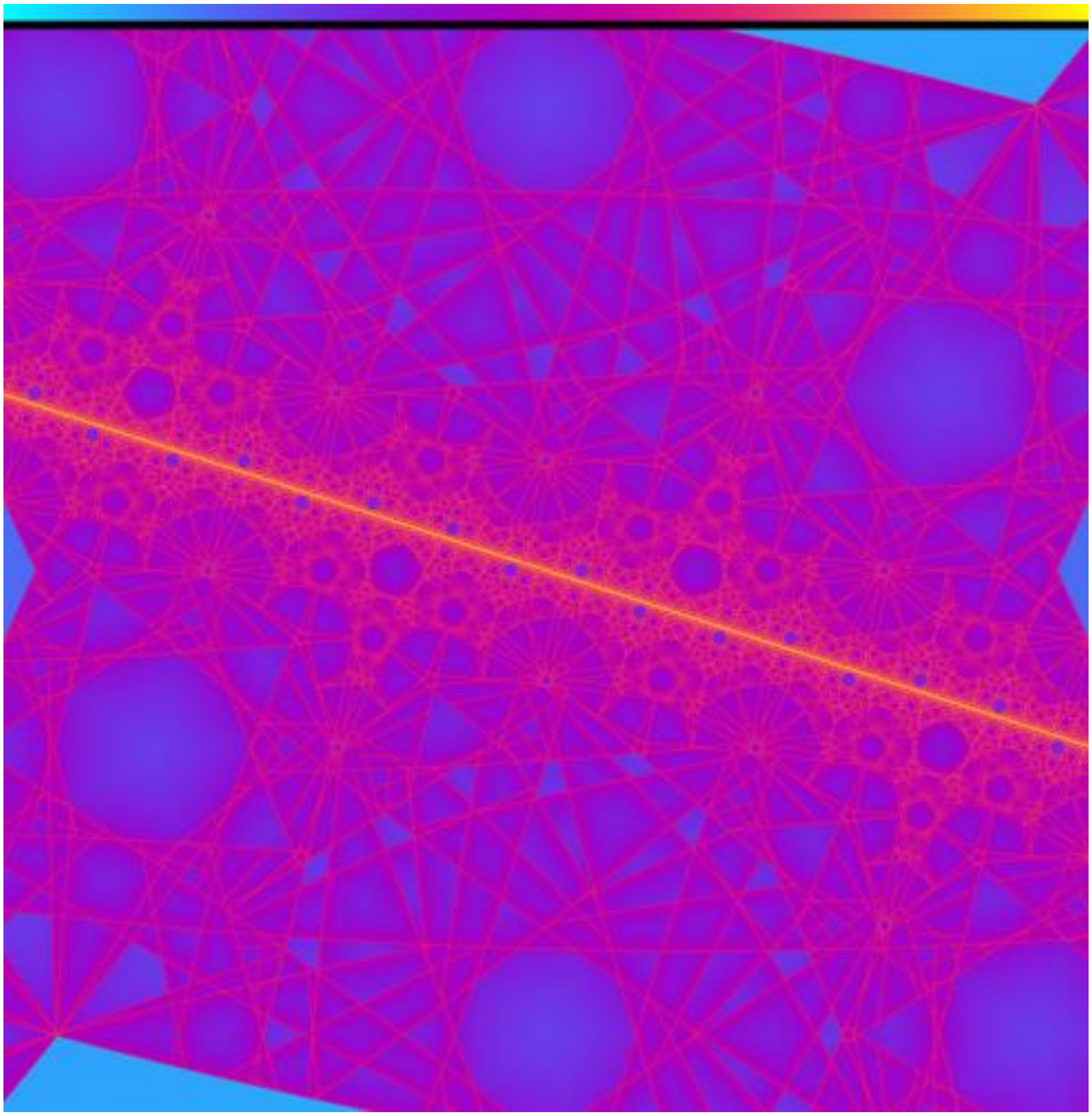
I'm not sure the border coloring tells us anything the absolute order coloring didn't

Here's N=8; R=1.71133; G=[A' B].” BE p10



Page 11 Thursday, June 2, 2016 thru Saturday, December 17, 2016

“I'm doing essentially the same calculation as before. I pick a point, cycle it over and over until it comes back around to where it started. Before I just tracked the number of A and B turns but now I also track the closest to the border the point got. The overhead of tracking the border proximity is rather low and I could make it much lower by smarter code re-using intermediate results.” BE, p11



"Here is a zoom on the origin of N=5; R=crit; G=[A' B] showing the critical line in more detail. I'm inclined to say the border proximity coloring doesn't tell us anything the absolute order coloring didn't already tell us." BE p11

BHearn wrote: "The Gathering for Gardner just uploaded videos of lots of talks from G4G11, 2014, including my talk about this. Carl had previously posted a video, but this one is higher quality."

https://www.youtube.com/channel/UCSOz6T52IsqvmcvHMQRxEgA

"I just finished watching this, and it completely explained everything that I couldn't understand about this thread. I've been subscribed to this thread for

about a year now, but I haven't completely understood everything, nor did I know the answer to the original question of whether or not the Gizmo Gear jumble. It's probably in here somewhere, but most of this thread has gone completely over my head. That was a great talk!
Now, forgive me for asking, but would you be able to elaborate on the method by which you generate these 2D puzzle images (or point images)? How does a computer systematically go through every possibility for the location of a point?
As well, is there a closed formula for the critical radius in terms of N? Or something similar?" k3DW, p11

"It's been a while since this thread was active but I wanted to share a slightly different avenue of exploration based on some newfound knowledge about numerical methods for finding algebraic numbers.

The idea is that if you have some floating point number, x, that you assume is algebraic, that means it's the root of some polynomial. If you can use linear algebra to find the linear dependence of x^0, x^1, x^2, ..., x^n and if you limit the precision of the calculation you may be able to find relatively small coefficients. The **PARI library has a function for this called algdep()** which I've been playing with.

So now that we know the critical radius for N=5 exactly, I think it's natural to ask what would it have taken to find it via numerical methods.
Bob **already mentioned this**." BE

BHearn wrote: "There is similar geometric reasoning to the above for N=5, leading here to R = sqrt(14 - 2 sqrt(5)) / 2 = 1.54336191843. I'll go through the details if anyone would like them. So, that's two critical radii we know exactly now.
Also, going back over my math from last night for N=5, it can be simplified with various trig identities to R = sqrt((7 + sqrt(5)) / 2) = 2.14896114175. So... yeah. If we had had enough digits we might have been able to find those expressions with algebraic solvers."

"So for N=5 we know R=sqrt((7 + sqrt(5)) / 2) which is one of the roots of x^4 - 7*x^2 + 11 and the decimal expansion looks like

2.148961141749634999249768705395459123897487959210140665278886210108784377308499999105421306990811116...

So what if we'd assumed the polynomial was quadratic and we'd had the first l digits of the expansion?

For l = 3 the best polynomial we find is x - 2 with root 2
For l = 4 the best polynomial we find is 3*x^2 + x - 16 with root 2.14874066490830075...
For l = 5 the best polynomial we find is 22*x^2 - 51*x + 8 with root 2.14896735183557126...
For l = 6 the best polynomial we find is 23*x^2 - 82*x + 70 with root 2.148963033616363418...
For l = 7 the best polynomial we find is 23*x^2 - 82*x + 70 (again)
For l = 8 the best polynomial we find is 129*x^2 - 119*x - 340 with root 2.14896112181419059
Obviously since the real polynomial is quartic, nod quadratic this isn't going to find the right polynomial and the wrong polynomial coefficients just get bigger and bigger the more digits we know.

So what if we assumed the polynomial was cubic?

For l = 3 the best polynomial we find is x^3 - x^2 - 2*x - 1 with root 2.147899035704787...
For l = 4 the best polynomial we find is 3*x^3 - 7*x^2 - 3*x + 9 with root 2.14910203397074...
For l = 5 the best polynomial we find is x^3 - 9*x^2 + 11*x + 8 with root 2.1489786617710881...
For l = 6 the best polynomial we find is 14*x^3 - 20*x^2 - 24*x + 5 with root 2.1489621997800...
For l = 7 the best polynomial we find is x^3 - 10*x^2 + 42*x - 54 with root 2.1489608575459549...
For l = 8 the best polynomial we find is 41*x^3 - 72*x^2 - 63*x + 61 with root 2.1489611407423782...

But the polynomial isn't cubic either. What if we assume correctly it's quartic?
For l = 3 the best polynomial we find is x^3 - x^2 - 2*x - 1 (again)
For l = 4 the best polynomial we find is x^3 - x^2 - 2*x - 1 (again)
For l = 5 the best polynomial we find is x^4 - 7*x^2 + 11 (CORRECT)
For all l > 4 we get the correct polynomial when search with l correct digits

So what if we'd assumed the polynomial was order 5?

For l = 3 the best polynomial we find is x^3 - x^2 - 2*x - 1 (again)
For l = 4 the best polynomial we find is 2*x^5 - 4*x^4 - 2*x^3 + 5*x^2 - 4*x - 1 with root 2.1489625947338441...
For l = 5 the best polynomial we find is 2*x^5 - 4*x^4 - 2*x^3 + 5*x^2 - 4*x - 1 (again)
For l = 6 the best polynomial we find is x^4 - 7*x^2 + 11 (CORRECT)
Until at least l = 100 we keep getting the correct polynomial

So what if we'd assumed the polynomial was order 6?

For l = 3 the best polynomial we find is x^3 - x^2 - 2*x - 1 (again)
For l = 4 the best polynomial we find is 2*x^6 - 3*x^5 - x^4 - 3*x^3 - 2*x^2 - x + 3 with root 2.14896387479987125...
For l = 5 the best polynomial we find is x^3 - x^2 - 2*x - 1 (again)
For l = 6 the best polynomial we find is 4*x^6 - 8*x^5 - 2*x^4 + 5*x^2 - 5*x + 3 with root 2.14896113118858765...
For l = 7 the best polynomial we find is x^4 - 7*x^2 + 11 (CORRECT)
Until at least l = 100 we keep getting the correct polynomial

So I think the moral here is that if we'd had at least 5 correct digits for R we could have found the right polynomial if we'd assumed it was quartic. We would have needed at least 6 digits to find it twice searching quartics or once with order 5 and 7 correct digits to find it three times.

So my feeling is that we would have needed at least 6 digits to have any confidence that R is a root of x^4 - 7*x^2 + 11
It's worth noting that the critical radius for N=10 is a root of the same polynomial. Curious." BE, p11

"Going off the above numerical methods for N=5, I've tried the same for N=7.
By brute force searching Bob thinks the lower bound for R with N=7 is 1.6235789 and the upper bound is 1.623579
If we assume the first 7 digits are correct 1.6235789 how close can we find a polynomial?

Order 2: 52*x^2 - 37*x - 77 with root 1.6235787497157328...
Order 3: 29*x^3 - 25*x^2 - 26*x - 16 with root 1.6235788957536696...
Order 4: 8*x^4 - 13*x^3 + 3*x^2 - 11*x + 10 with root 1.62357885102866068...
Order 5: 9*x^5 - 4*x^4 - 11*x^3 - 4*x^2 - 5*x - 8 with root 1.6235789071387607...

Order 6: 2*x^6 - 2*x^5 - 3*x^4 + 2*x^3 + 7*x^2 - 10*x - 4 with root 1.6235789148295022...
Order 7: 4*x^7 - 4*x^6 - 2*x^5 - x^4 - x^3 - 2*x^2 - x - 5 with root 1.6235789045589750...
Order 8: 2*x^8 - x^7 - 4*x^6 + 3*x^5 - 3*x^4 - 2*x^3 + 2 with root 1.623578900664842628...

If instead we assume the first 6 digits are correct 1.623579

Order 2: 52*x^2 - 37*x - 77 (again)
Order 3: 12*x^3 - 22*x^2 - 7*x + 18 with root 1.6235804669659...
Order 4: x^4 - 6*x^3 + 14*x - 4 with root 1.62358128684672...
Order 5: x^5 - x^4 + 4*x^3 - 7*x^2 - 3 with root 1.6235798460588...
Order 6: x^2 - x - 1 with root 1.618033988749...
Order 7: x^2 - x - 1 (again)
Order 8: x^2 - x - 1 (again)

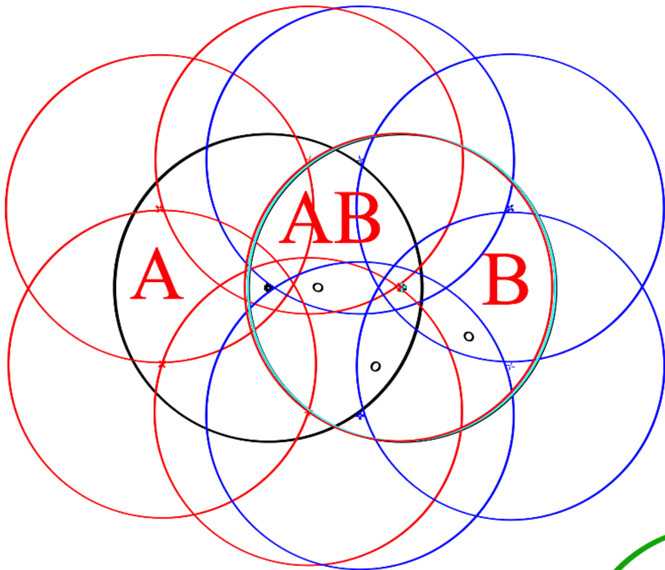
So if the critical radius for N=7 is algebraic (and it might not be!) we need more digits to find the closed using numerical methods." BE, p11

End of posts **Sat. Dec. 17, 2016**

Are the Gizmo Gears jumbling?

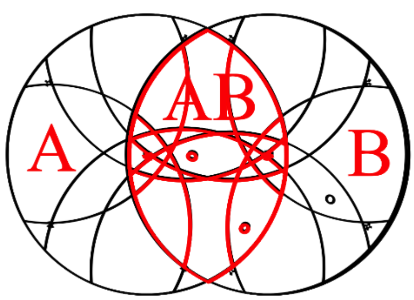
YES!

End Notes



Demo of an N = 5 using whole circles.

Here is a simple set of rotations of the AB intersection with N=5. We first do only 4 rotations, or 4A then 4B of the A, B circles, as seen in the top diagram. It creates about 15 basic pieces, at bottom left. As with the surd clock mentioned earlier it has bilateral symmetry. Rotate the AB intersection



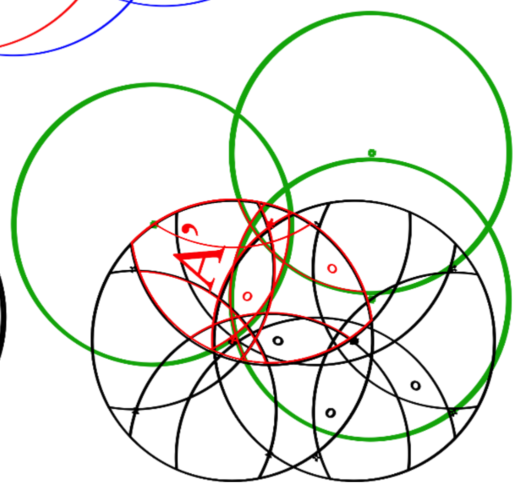


Figure E1

by doing A' to where three circles make new cuts, as you can see in the bottom right diagram. You can see that new cuts will be required as you continue doing A'. Thus, the system will remain bilaterally symmetric if we continue doing [[4A'], 4[B]]. This shows how quickly the circles cut things up into smaller pieces. The disadvantage is that it might not produce the beautiful generator fractals like Bob Hearn's, Brandon Enright's, and Jason Smith's. On the other hand it could produce fractals if there were algorithms to light up certain points or pieces. Also there may be some areas that remain uncut. I am not sure if this is any different than the way that Bob

Hearn generated his initial full circle fractals. You can see that this process will distribute the central AB symmetry around as the center wedge rotates thru full cycles. New cuttings will occur inside the AB wedge as it rotates. There will be a small area in the center of AB that remains uncut because R is greater than 2. This was discussed often in the forum.

Using whole circles shows how set theory might be applied to gizmo gears type puzzles. Each circle is part of a Venn diagram. Sets are rotated into and out of each other and leave their traces when generating fractals. You could have erasing circles as well that decrease the number of pieces. The intersection set is the basic piece shapes and their number of occurrences as elements of any group of circles, where pairs of circles are easiest to deal with.

The sets could be more varied by adding colors to the pieces, based on orientations, coordinates, sequential occurrence (odd, even, etc.), and so on. By superimposing one puzzle on top of another the colors become color mix sets similar to the pixels on a graphic screen. You could create a color and shape changing fractal as a screen saver and so on. This is similar to my Space Pixels game here: https://www.puzzleatomic.com/GAMES_4.htm .

A really cool looking fractal can be generated by only putting a bright colored point at the center of each new circle. This will result in a projection of the generator fractals discussed in the forum. It will show up as a graphic both inside the two A, B circles and surrounding them. If it were limited to only appearing outside of the A, and B circles it would form a cloud around the standard Gizmo fractals.

Conclusions, Comments, Questions

Bram Cohen proposed to investigate the idea of jumbling twisty puzzles.

The Gizmo Gears puzzle jumbles was discovered by Bob Hearn.

Many new crystal-like fractals have been produced. These may have use in art, math, design, entertainment, and possibly, someday, in chemistry (crystallography) and physics, who knows?

All the N=5, N>=7 produce fractals at or greater than a critical radius, Rc, where Rcx is >= Rc. Rc and Rcx do not apply to N=2,3,4,6.

Simple repeating generators form unbandagings at some Rcx.

For N = 5 Penrose tiles can be simulated, found by Bob Hearn.

Jason Smith and Brandon Enright produced the first interesting simple generator fractals.

Brandon Enright produced a mathematical analysis of Gizmo fractals Rc by finding some of their closed form polynomials and concluded that some N could possibly require higher order math to solve for Rc.

An open problem is to produce fractals with full circle symmetry such that both A and B circles are N rotationally symmetric (no angled bars cutting across the center, and each circle looking identical). Perhaps Bob's code.

Mathematicians should do a careful analysis of twisty puzzle fractals to determine where their mathematical fit is as applied to set theory, geometry, combinatorics, crystallography, topology, number theory, information theory, cryptology, etc. Should these fractals be a new branch of math?

A simplest simple generator that produces a fractal at the smallest N and R for two equal circles should be found, or at least proven.

Infinite repeating tile fractals should be investigated. See Roice Nelsons Magictile.com game system. Or is this even possible?

The Cardinality of the circle fractals for different situations should be formulated and proven.

Bob Hearn's method of cutting into distinct pieces should be implemented to see what happens to a beginning patterned symmetrical coloring of the two circles as the fractal develops. This would be similar to mixing the colors of a Rubik cube using a simple generator. It should produce interesting results of symmetrical pattern mixing. A kind of paint mixing art.

Blank page for notes: