Integer Code For Linked Circles

Presenting some new integer invariants

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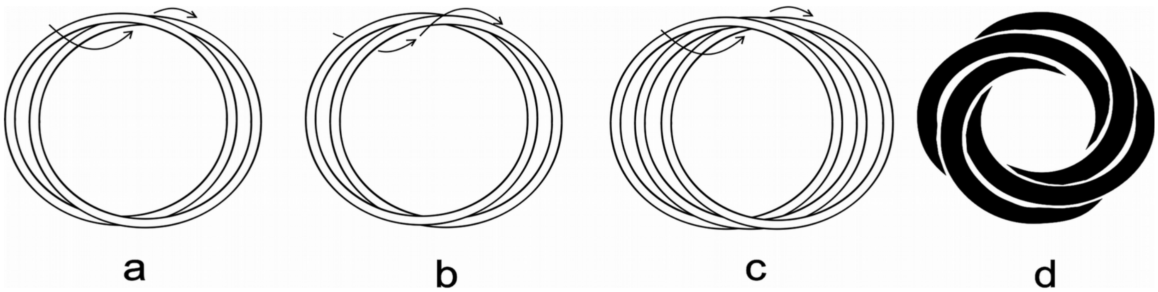
If c_n is the number of distinctly different ways of linking n identical circles so that every pair is linked, then as n increases c_n becomes more difficult to determine. Restricting links to be circles might seem to make this a simple exercise but c_n rapidly increases and also the complexity of the possible linkages increases as n increases. In addition the twist can increase dramatically for each added link when n grows large. The regularity of using identical circles produces a hierarchy of different kinds of structures as n increases that make the system mathematically interesting. It fits a mathematical way of thinking abstractly. For instance any linkage can be thought of as a single circle so that a given linkage can can be linked to another linkage also thought of as a single circle.
In a search of the literature no formula for c_n was found. If the circles can be arranged so that their centers coincide with a line then the linkage is **linear** and there is a simple integer construction code, **CC**, for constructing it. This code will also allow operations on the code such as linear rearrangements and **twist** calculations for any linear arrangement. When you get to a linkage of six or more circles it is possible to make the linkage in such a way that a linear arrangement is not possible. This is called a **Rogue link**. There are probably exponentially more Rogue links than there are linear links as n increases because a binary flat linear projection is not possible.

Define '**All circle links**' $\mathbf{A_{CL}}$={} as the set of equal sized linked torus(circle) shapes. Each torus has major and minor radii **R** and **r** and major and minor diameters, **D** and **d**. We can make the linked circles thinner as needed until d/D becomes 0 in the limit, equivalent to a geometric circle. If d/D>0 only a certain maximum, n, of these circles can be linked so that every circle links thru every other circle. For small values of n this maximum has an approximate value of **n=(2D/d)-2**, so n can be made larger by making d smaller or D larger. This is because as n increases there is less and less room for n-1 circles to pass through a single circle due to their thickness, or small diameter d.

A **regular** $\mathbf{A_{CL}}$= of n links is one where each circle links through **q** circles and does not link through **k** circles thus k+ q =n. A a circle cannot link to itself so n>= k>=1 and 0<=q<=n-1. We will only consider the case where every pair of circles is linked so k=1 and q=n-1. Since k is constant we have the set description $\mathbf{A_{CL}}$={k1}. The well known Hopf link consists of two linked circles where each circles d center passes through the D center of the other circle at right angles. This two circle link is the simplest non trivial link[5]. However $\mathbf{A_{CL}}$={k1} circles need not pass through each others centers.
For linear $\mathbf{A_{CL}}$={k1} and if the circles are thin enough you can lay them out on a surface in a projected **flat linear array, L**, defined as having the center of each circle lie on a straight line. Thus we add L to the set of $\mathbf{A_{CL}}$={k1,L} links we will be concerned with.

A simple integer construction code CC describes the exact construction of any $\mathbf{A_{CL}}$={k1,L}. Define a **close link** as a linkage of two or more circles that are linked so that each circle twists in the same direction around the other circles. We use twist in place of linking number because circles cannot bend like a ribbon so writhe = 0. Twist calculations for a link projection are defined in knot theory[1]. Each pair of circles in the close link can be manipulated and made to touch around their circular torus surface. Five or more circles can be made where no close links are possible, called a **prime link**. Thus **5 prime**, **5p** is the smallest possible prime link.



$\mathbf{A_{CL}}$**={k1,u0,m1,n2,L}, twist = +1** **twist = -1** $\mathbf{A_{CL}}$**={k1,u0,m1,n3,L}, twist = 3**

Figure 1

Figure 1, a and b shows two linked circles laid over or projected in two ways as close links. The pair on the left have a construction code of CC=1,2 while the pair on the right have CC=2,1. Links of a close link can be seen to twist in as shown in Figure 1. The smallest close link is two linked circles. The two circles in Figure 1, a have one positive twist while the two circles of the identical linkage in Figure 1, b have one negative twist [1]. If more circles are added to a close link the twist becomes more 'locked in' but can still vary by laying the circles over in different ways. Close links of n links allow the n circles to be placed in any marked order or n factorial possible ways. The total twist is maximum when the entire link of n circles is a single close link in a plane or flat projection. A simple way of calculating the twist from its CC coded linear flat projection $\mathbf{A_{CL}}$={k1,L} will follow below. For a linkage of n circles **u** is the total number of circles that cannot be made close to any other circle.

Close links of different twists can be linked together in various ways. This variable is called **m** and is the number of different close links in $\mathbf{A_{CL}}$={k1,L}. For instance you could have a link where 5 close links twist one way linked to another where 3 close links twist the opposite way making m=2, u=0. It may seem that u and m are redundant used together, but for many links they are not redundant. It is evident that if m=0 then u=n. This is called a prime link, mentioned above, where no pair of links are close(cannot be made to come close by manipulation of the circles). We also have the total twist of the flat projection, 0, +t or -t. All of these variables, including twist, will not guarantee a unique linkage. This becomes more prominent as n increases. By folding each circle of $\mathbf{A_{CL}}$={k1,L} over for a cycle of n folds n different layover arrangements occur completing the cycle. This produces n rows of n integers per row which is a **CC matrix**. The twist for each linear fold arrangement is summed per row then the row twists are summed for a matrix total twist M_{TT}. The M_{TT} is always a constant or invariant for that linkage for all possible unique CC coded folding matrices.

Figure 2 top left and right shows a **linear 5 prime linkage** of five circles. The bottom left shows the result of one fold of the top left CC while the bottom right shows the **fold order, FO** this creates. Five prime is the smallest possible prime so that u=n=5. Refer to the top left part of Figure 2 to see how a **construction code**, **CC**, is written for this linkage. The leftmost circle is in the level 5 z axis position. The next to leftmost circle is in the level 2 z axis position and continuing we can write the level, or z axis position of each circle moving from left to right along the x axis as $5_1,2_2,4_3,1_4,3_5$. The FO code is a CC that represents the **fold order, FO** of its generating CC. Derive FO from CC as $\mathbf{p_q = q_p}$ ie. $\mathbf{CC(p_q)=5_1,2_2,4_3,1_4,3_5=FO(q_p)=4_1,2_2,5_3,3_4,1_5}$ (the position subscripts are normally not written). Fold order is just a CC gotten from its generating CC. The FO specifies the order of CC circle folding about the x axis where the FO= 4,2,5,3,1 takes the left most 4 to mean fold the CC circle in x position 4 forward and down about the x axis to generate the next row of the CC matrix. Thus 5,2,4,1,3 become 4,1,3,5,2, The CC code is circularly permuted along the z axis.

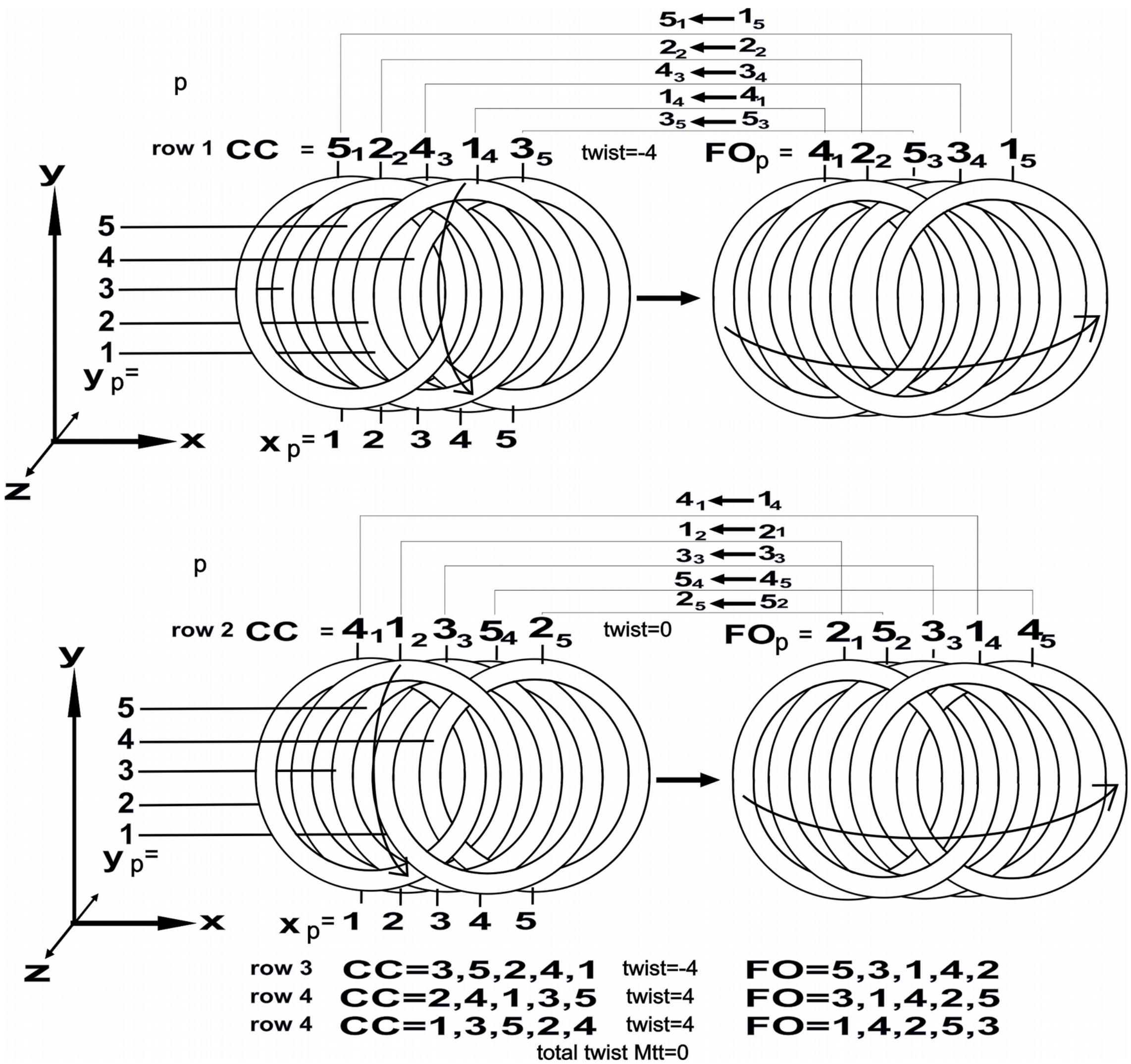


Figure 2

Normally it is a convention to put the left CC circle in the level 1 position when writing the first row of a CC matrix. The convention is not followed here to show generality. Level one is always the first circle that can be folded downward for a new layover position as shown by the curved arrow on the top left group of circles.

The **fold order**, or **FO** is written from the CC based on going along the x axis and writing the leftmost x position of the first circle in z level 1, then write the next x position of the circle in z level 2 and so on. Thus every CC has an FO and every FO can be exchanged with every CC to generate new matrices. The CC and FO codes are exact complements. The FO is the order of folding of the CC circles when folding each circle one at a time down or toward you. First fold down(about the x axis) the circle in position four (CC code=1 FO code =4) then the next circle able to fold is in position two(CC code=2, FO code =2) and so on. You generate the next FO row by rotating the leftmost FO circle about the y axis so that it becomes the rightmost circle. Thus 4,2,5,3,1 become circularly permuted to 2,5,3,1,4.

A prime requires that u=n so that no pair of circles can be made close to each other. For a prime any two adjacent integers in the construction code or CC must have a difference greater than 1. You can pull the circles apart in two groups oppositely along the y axis to put the linkage into a position where CC and FO exchange places. Figure 4 shows this process for a 6 prime linkage. Pull the left upper 3 links left and the right upper 3 links right then push the linkage top down and bottom up to form the horizontal arrangement. Because no folding needs to take place to do this the **twist of CC always equals the twist of its FO**, (all pairs of circles remain in the same twist relationship) and CC and FO can exchange places enabling the full complement of all the 2n possible folding matrices.

The twist of a linear $\mathbf{A_{CL}}$={k1,L} can easily be calculated from the CC. Twist can vary depending on how the circles lay over each other in pairs. The twist for a given CC row then is the sum of the twists of each left-right ordered pair of circles. Because the linkage is linear twist can be calculated by looking at all possible left-right ordered pairs of numbers in the CC or the FO thus:

twist(CC,L) = sum [(CCp_iCCp_j) of all pairs i<j and $CCp_i<CCp_j$: t = +1 and $CCp_i>CCp_j$: t = -1]
twist(FO,L) = sum [(FOp_iFOp_j) of all pairs i<j and $FOp_i<FOp_j$: t = +1 and $FOp_i>FOp_j$: t = -1]

For instance the numbers a,b form exactly one ordered pair, a,b, while a,b,c produce 3 ordered pairs a,b, a,c, b,c. Thus a linear CC of n numbers produces n^2/2 – n/2 ordered pairs. Each ordered pair has twist of +1 if a<b and twist of -1 if a>b. The calculation for 5 prime CC=1,3,5,2,4 then results in t(1,3,5,2,4) = 4 The calculation for the 5 CC rows of the matrix formed by doing one fold for each new row results in a total matrix twist of zero for 5p. Five prime is its own mirror image but many random CC codes can have a zero total matrix twist.

The twist calculation for a linear $\mathbf{A_{CL}}$={k1,L,n} works for any random arrangement of the positive sequential numbers 1 thru n.
Note that for certain number sequences CC sometimes exactly equals FO. This is the case for CC=1,4,6,2,5,3 = FO=1,4,6,2,5,3. For the nxn code matrix row i twist of CC equals row i twist of its FO. CC and FO are complementary operators where and are twist equivalent being simple rearrangements of the circles without folding any of the circles over.

The CC or FO code allows any simple list of sequential integers to be rewritten in 1 thru n form(ex 6,4,5 would be 3,1,2) to analyze the number pattern in comparison to other patterns along with their twists.

Some Rules for a linear $\mathbf{A_{CL}}$={k1,L} to be prime:
1. No close links are allowed so that code numbers a_i satisfy $n-1>|a_i-a_{i+1}|> 1$ and $n-1> |a_1-a_n|> 1$, necessary to prevent close circles.
2. $\mathbf{A_{CL}}$={k1,L} is not a composite prime, ap#bp necessary because thought of as circles they would be be a pair of close circles.

A method for building an n+1 prime from an n prime:

Put the n prime linkage in a position where left and right code numbers are not equal to 1 or n:

$1<CC_1<n$. $1<CC_n<n$.

This brackets the two ends with circles that are not ready to be folded(not in level 1 or level n). Now add a link to one end (a link through the common center of the existing circles). This locks in the new linkage as n+1 prime, (n+1)p.

Five prime has only one position like this available to make 6 prime. Thus only one 6 prime is possible.

Linear Permutation Matrices `

An nxn matrix is our goal. To generate the next row of a CC matrix of n links the process is:

if ($CC_{1,j}$) = row 1 = 1,5,2,7,3,6,4, then (row 2 - 1) = 0,4,1,6,2,5,3,

and change 0 to n since the 1 position is always folded over to become the n position,

producing row 2, ($CC_{2,i}$) = 7,4,1,6,2,5,3

To generate the next row just subtract 1 from each number in the row vertically above. If the number is 1 then change it to n.

For FO fold order matrix you can generate each FO row from its corresponding CC row, as already shown above in Figure 2. This is the method for generating the first row of the FO matrix. The following rows can be generated this way as well.

Alternately you can generate rows 2 and greater by circularly left permuting the FO numbers of the row position in the row above.

Here is an example matrix for a 7_p with CC=1,5,2,7,3,6,4.

CC, FO n row matrix

7_p LP1=1,2,3,4,5,6,7(marks on the circles) 7_p LP2=1,3,5,7,2,6,4(new mark order CC1 to CC2)

7p CC1	twist	7p FO1	twist	7p CC2	twist	7p FO2	twist
1,5,2,7,3,6,4	+7	1,3,5,7,2,6,4	+7	1,3,5,7,2,6,4	+7	1,5,2,7,3,6,4	+7
7,4,1,6,2,5,3	-5	3.5.7.2.6.4.1	-5	7,2,4,6,1,5,3	-5	5,2,7,3,6,4,1	-5
6,3,7,5,1,4,2	-9	5,7,2,6,4,1,3	-9	6,1,3,5,7,4,2	-1	2.7.3.6.4.1.5	-1
5,2,6,4,7,3,1	-5	7,2,6,4,1,3,5	-5	5,7,2,4,6,3,1	-9	7,3,6,4,1,5,2	-9
4,1,5,3,6,2,7	+7	2,6,4,1,3,5,7	+7	4,6,1,3,5,2,7	+3	3,6,4,1,5,2,7	+3
3,7,4,2,5,1,6	-1	6,4,1,3,5,7,2	-1	3,5,7,2,4,1,6	-1	6,4,1,5,2,7,3	-1
2,6,3,1,4,7,5	+7	4,1,3,5,7,2,6	+7	2,4,6,1,3,7,5	+7	4,1,5,2,7,3,6	+7
M_{TT} total	+1		+1		+1		+1

Generating the 14 unique matrices for 7p

The matrix total twist, (M_{TT}) is always the same. This means that M_{TT} is always an invariant constant for any CC or FO matrix for any specific linear $\mathbf{A_{CL}}=\{k_{1,L}\}$. A total of 2n **different matrices** are possible for a linear $\mathbf{A_{CL}}=\{k_{1,L}\}$. For a prime such as 7p that has no symmetry and is not its own mirror image each CC matrix generated from different FO rows used as CC starting rows will produce columns of row twists that are different, but the M_{TT} totals for each matrix are equal. For a link that is a single close link there is only one matrix because all 2n matrices are alike, just having rows in a different permuted order.

Each of the two fold order matrices, FO1 and FO2 has n rows each of which can be used as the first row for generating a new CC matrix of n rows. These 2n CC matrix can be called CC1,1, CC1,2, … CC1,n and CC2,1, CC2,2, … CC2,n. If a single nxn matrix of these columns of row twists for a set of n matrices is built, each row and each column of twists will add to the same total twist, M_{TT}. Below is a matrix of these n matrices with the n row twists built for the FO1 starting row of the 7_p matrix. Examination shows all rows to be unique with no circular permutations of other rows.

FO1 starting CC rows creating n 7x7 twist matrices each with 7 row and 7 column Matrix total twist numbers with 7 twists per row and column. This is summary of the n different CC1 matrices.

		M_{TT}(matrix totals)
CC1,1 row twists	7,-5,-1,-9,3,-1,7	1
CC1,2 row twists	-5,7,7,-5,3,-5,-1	1
CC1,3 row twists	-9,-1,-5,7,11,-1,-1	1
CC1,4 row twists	-5,-1,-9,-1,-1,11,7	1
CC1,5 row twists	7,7,-5,-1,-5,3,-5	1
CC1,6 row twists	-1,-5,7,7,-1,3,-9	1
CC1,7 row twists	7,-1,7,3,-9,-9,3	1
Column totals	1, 1, 1,1, 1, 1,1	

FO2 starting CC rows creating n 7x7 twist matrix
each with 7 row and column Matrix total twists with 7 twists per row and column.

		M_{TT}(matrix totals)
CC2,1 row twists	7,-5,-9,-5,7,-1,7	1
CC2,2 row twists	-5,7,-1,-1,7,-5,-1	1
CC2,3 row twists	-1,7,-5,-9,-5,7,7	1
CC2,4 row twists	-9,-5,7,-1,-1,7,3	1
CC2,5 row twists	3,3,11,-1,-5,-1,-9	1
CC2,6 row twists	-1,-5,-1,11,3,3,-9	1
CC2,7 row twists	7,-1,-1,7,-5,-9,3	1
Column totals	1, 1, 1,1, 1, 1,1	

The nxn matrix represents a completed fold cycle about the x axis for each matrix. The column twists have the same total because they start with the next FO of the first FO matrix as the first CC of the next matrix. Each matrix represents a 180 degree rotation of the other matrix about the top left to bottom right diagonal and are equal in this sense: CC2 vertically equal to CC1 horizontally and vice versa. This makes sense because CC proceeds by folding about the x axis and FO proceeds by folding about the y axis, a 180 degree rotation of the complementary twist matrix results. Thus instead of 2n, only n matrices are essentially different, but without exchanging CC and FO to generate new matrices the n matrices would not be as straightforward to find.

Each of the row twists is an odd number or an even number for a given linkage of n links(zero included as even). If the number of additions to calculate a row total twist is odd and n is odd then the matrix total twist for that linear $\mathbf{A_{CL}}=\{k_{1,L}\}$ cannot be zero. Thus if n=3, n=7, n=11 prime or any n=3 + 4v where v=0,1,2... cannot have a zero M_{TT} and must have a minimum of +1 or -1 M_{TT}.

Mirror Image and Axial Rotation

A mirror image CC code is designated with an underline as $\underline{CC}$. The mirror image of a CC code is the code written in reverse thus **mirror image of $\underline{CC}$ = $CC_{((n+1)-p)}$ (ie. reverse positioning by writing the $\underline{CC}$ code in reverse of CC ie. left to right becomes right to left).** Rotation about the x,y,z axis is designated CX, CY, CZ and leaves twist unchanged. CX does not change the code but reverses the binary marking or coloring of the circles(ex. flat circles with black on one side and red other side). CY changes the code and reverses binary marking. CZ changes the code to the same code as doing CY but retains binary marking so we shall use it to derive distinct matrices. Then:
CZ = $((n+1) - CC)_{((n+1)-p)}$ ie. subtract each number of CC from (n+1) then reverse left to right positioning to get the CZ code. The mirror **$\underline{CZ}$ = (n+1) - CC** which is the same as:
$\underline{CZ}$ = $CZ_{((n+1)-p)}$.
The twist for a mirror code is the negative of its code, thus
t(CC) = -t($\underline{CC}$),
t(CC) = -t($\underline{CZ}$).

Any $\mathbf{A_{CL}}=\{k_{1,L}\}$ can be added to another $\mathbf{A_{CL}}=\{k_{1,L}\}$. We can add five prime to its mirror, 5p#$\underline{5}$p, to produce a composite prime. Think of 5p as a single circle then link another 5p circle to it. The code of for this can be made by writing the 5p code then appending $\underline{5}$p + 5 code to it on the right or left(add n to each code number). Thus:

1,3,5,2,4,9,7,10,8,6 = 5p#$\underline{5}$p with Mtt = 0, also,
1,3,5,2,4,10,9,8,7,6,16,17,18,19,20,14,12,15,13,11 = 10n#$\underline{10}$n with Mtt = 0 because the last 10 integers are the mirror image of the first ten integers. An additional property of mirror image composition is if the added codes are interleaved then Mtt = 0. Thus 1,3,5,9,2,7,4,10,8,6 = t(5p#(interleave 1, $\underline{5}$p))=0. Thus interleaving adds up to n-1 new linkages and unique matrices. Interleaving has been found to work only for exact mirror composition, not for FO#$\underline{CC}$, or $\underline{FO}$#CC composition combinations.

If we add a CC to its mirror image, $\underline{CC}$ we always get Mtt(CC + $\underline{CC}$) = 0. By composing 5 prime with its mirror image in several different ways we get 16 unique matrices for n prime, 2 for each combination. We have:

CC#$\underline{CC}$, **$\underline{CC}$#CC,** we also have $\underline{CC}$#FO, FO#$\underline{CC}$, CZ#$\underline{FO}$, $\underline{FO}$#CZ, $\underline{CZ}$#FO, FO#$\underline{CZ}$,
CC#$\underline{CZ}$, **$\underline{CZ}$#CC,** CC#$\underline{FO}$, $\underline{FO}$#CC, CZ#$\underline{FZ}$, $\underline{FZ}$#CZ, $\underline{CZ}$#FZ, FZ#$\underline{CZ}$,
$\underline{CC}$#CZ, **CZ#CC,** CC#$\underline{FZ}$, $\underline{FZ}$#CC, $\underline{CC}$#FZ, FZ#$\underline{CC}$,
$\underline{CZ}$#CZ **CZ#$\underline{CZ}$,**
where all CC and CZ, FO, FZ represent identical linkages(but in different code positions). All Mtt = 0. More codes can be added such as **CC#$\underline{CC}$#CZ#$\underline{CZ}$ and CC#$\underline{CC}$#$\underline{FO}$#FO** for Mtt=0 by being careful to alternate added codes.
Composition of primes by thinking of each prime as a single circle allows more abstraction such as squaring a prime by composition. For instance 5p#^2 would be linked as (5p+(0*5))#(5p+(2*5))#(5p+(4*5))#(5p+(1*5))#(5p+(3*5)) (multipliers work out as code adders. 0,10,20,5,15, ie 5p *circles are linked as* **1,3,5,2,4** which is the code for 5p) meaning that each prime is laid over so that no two of the five 5p *circles* can be close.

Adding rows to produce zero twist
If Mtt is zero and two rows of a CC or FO matrix have equal but opposite twists, one positive and the other negative, then adding the two rows to produce a composite linkage always produces zero Mtt for that combined linkage.

Linkage Total Twist, L_{TT} and Link Total Twist, L_{KT}
For linear, or $\mathbf{A}_{CL}$={k1,L} The matrix total twist, M_{TT}, is the sum of all possible twists for a linear code cycle. Each matrix has n rows so dividing by n gives the linkage average twist $L_{TT}=M_{TT}/n$.

Each circle can then be given an average twist called the link average twist(twist per individual circle), $L_{KT}= M_{TT}/n^2$ or L_{TT}/n.
When we calculate the M_{TT} twist for a max twist linkage, where every link twists in the same direction (a close link of n links) we find that as links are added the twist added as n increases tends to rise slowly toward a straight line. Twist data can be gotten using the matrix calculation methods detailed in ref. [6]. The L_{KT} for max twist(close) linkages n=10 to n=20 links is:

10	11	12	13	14	15	16	17	18	19	20
1.2	1.36	1.52	1.69	1.86	2.02	2.19	2.35	2.5185	2.6842	2.85

For $\mathbf{A}_{CL}$={k1,L} The increase in twist per circle going from n=15 to n=16 is approximately 0.1653 and this increment between n=19 and n=20 is 0.1658 or an additional increment of 0.0001 when adding 1 link then link average twist increases as $L_{CT}=0.16n$ approximately for this example, tending to a straight line increment. If this link twist continues to increase at a small but steady pace it must eventually become large for each circle in the linkage as n gets large. Each added link would add considerably to the L_{TT}, linkage total twist. Each link of a 10000 link A_{CL}={k1} would add about L_{KT}=0.16*10000=1600 twists per per link when adding one link. Therefore total linkage twist, L_{TT}=1600*10000= 1.6x10^7 approximately.

Binary Marking
Figure 3 shows a linear 6 prime with a binary marking. The circles are colored perpendicular to the large radius R, light gray on one side and white on the other side. Twist is based on the ordered pairing of the circles. This means any rearrangement that does not involve folding or turning any circles over after rearranging will have identical total twist after rearranging since it has the identical set of ordered pairings.

The binary marking uses powers of 2^0 thru 2^(n-1). One side has the power numbers and the other side has the identical power numbers with underlines meaning a mirror image number. We use underlines since a circle is its own mirror image. This allows computing all pair configurations as the sum of the smaller of images or mirror image numbers where $\underline{a}$+b=a+$\underline{b}$=2^(n)-1. For instance all underlined on one side means all not underlined on the other side both sums are 2^(n)-1 so we take the sum as zero, ie. a+$\underline{b}$ = 0 + $\underline{2\text{^}(n)\text{-}1}$ and 0 is the smaller number. For only number $\underline{1}$. underlined the sum is 1 since $\underline{1}$+126=127=a+b. The maximum possible number of unique binary sums is thus (2^n)/2, or for 7 prime 2^6=64. We say possible because not all these sums are possible with the CC matrices. Only up to n unique matrices are possible. For 7 prime, n=7 matrices with 7 rows per matrix we have n^2 = 49 possible unique sums but in practice several of these rows will be the same binary power sum.

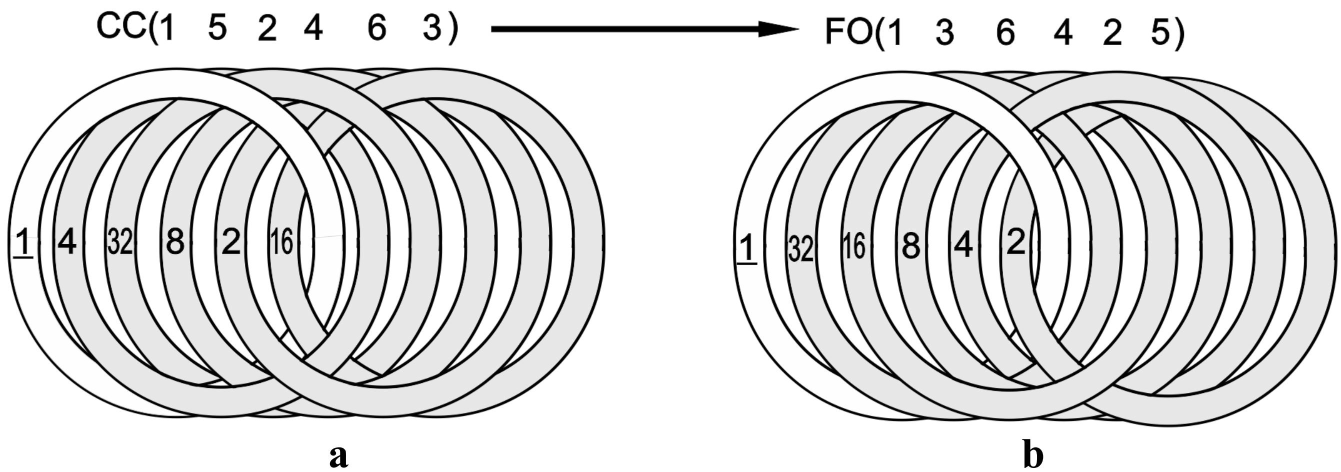


Figure 3

Note that we do not care about the marked order of the sums since the circles can be marked with the powers of 2 in any random order. We care only about the number of possible different sums since each different sum represents a different way of laying over the circles, ie. their ordered pair coupling arrangement.

Demonstration of t(CC),=t(FO), Twist Invariance for A_{CL}=$\{k_{1,L}\}$

It is necessary to rearrange the CC and FO so that they exchange places with each other without folding

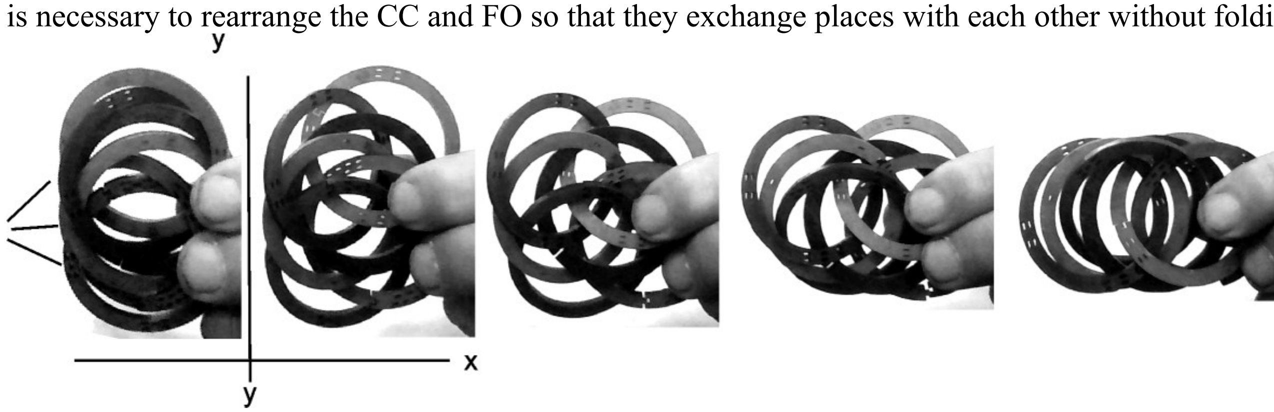


Figure 4

any of the circles, CC becomes FO which is itself now a CC(or an FO for its CC). Refer to Figure 4 for an illustration of a way to do this. Grasp the uppermost(z axis) links with your fingers on left and right of the y axis line of links and pull them apart along the x axis and then move them together along the y axis thereby transferring the linear array of circles to line up along the x axis. Because of the way CC is generated by writing the z level along x and FO is generated by writing the x level along z then CC and FO can change places with each other without changing the ordered pair arrangement, ie. without any folding over of any of the circles. Since this operation does not involve any folds of the circles the twist of CC must exactly equal the twist of FO. This will work for a linear linkage of any size n but would rapidly become unwieldy as n increases. It shows that a simple physical method exists in addition to the mathematical code method of analyzing position changes of the code numbers CC to FO to prove the ordered pairs sums are equal, t(CC)=t(FO).

Matrix Total Twist Invariance , M_{TT}, for A_{CL}={k1,L}

The set of folds that generate each row of a CC matrix always complete one cycle after n folds generating n rows of the nxn CC fold matrix. This holds true for all linear linkage codes A_{CL}={k1,L}. Each pair of circles have two layover positions, shown in Figure 1a and 1b. No matter how the circles are rearranged in a line to begin a fold matrix a complete set of n folds will produce a complete set of pair comparisons since folding can only occur as a simple permutation process. The entire set of ordered pair comparisons for each nxn matrix must always produce the same sum making the matrix total twist invariant for all linear arrangements of any A_{CL}={k1,L}.

Toroidal Rotation for Linear A_{CL}={k1,L} Links

It has been found that if a A_{CL}={k1,L} linkage is made of thin circles and held flat in a circular and symmetrical arrangement it can appear to turn inside out about its toroid axis. The circles all move simultaneously to opposites sides of the surrounding torus space. This has been found to work for some primes and works well if the n circles are a single close link. It is conjectured to be similar to the process of changing CC to FO as in Figure 4 where CC to FO would be isomorphic to a ½ toroidal rotation. Of course with more and more circles we need them to be thinner and thinner. The twist remains the same as the linkage rotates, meaning that it is always in the same flat arrangement even though the circles are moving about with respect to each other.

Symmetrical Node Puzzles for Linear A_{CL}={k1,L}

It is possible to arrange an A_{CL}={k1,L} linear link in a symmetrical flat circle. By passing symmetrical

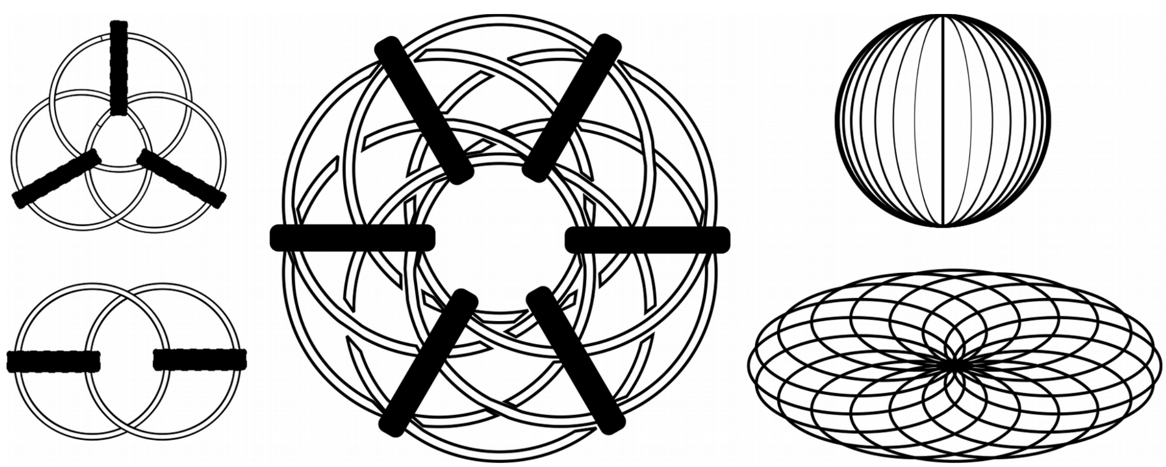


Figure 5

groups of node crossings thru holes made in some bars, a puzzle like device can be made. The simplest, which I call diaka,(in sequence 3=triaka, 4=tetraka, 5=pentaka, etc.) consists of only two

circles and does not use any node crossings. For more than two circles, if the holes in the bars are big enough to allow for some clearance and are spaced just right then you can rotate the bars continuously in a group about a circular torus axis. When doing this rotation the circles appear to turn inside out too, but instead they just move from side to side through the torus space thereby making a nice optical illusion puzzle. The motion seems hard to imagine, especially for puzzles with a greater number of circles n. Of course the 6 circle structure shown, at the left in Figure 5 (called 'Hexaka') cannot be perfectly flat when in the flat position because the circles have thickness and pass over each other at the nodes. When the bars rotate 90 degrees about the toroid axis the figure emulates a spherical shape with holes at the poles. The idea can be extended. Imagine that a large number of very thin $\mathbf{A}_{CL=}\{k\mathbf{1,L}\}$ circles are placed closely in a flat circular array as seen at lower right in Figure 5. If some method of holding the nodes symmetrically existed this circle shape could be folded into a spherical shape as seen at top right in Figure 5. Continuing the folding would return it to a flat circle shape. The flat circle shape has a point in the center where a concentration of all the circles pass over each other while the spherical shape has two opposed polar nodes where this happens.

Non Linear $\mathbf{A}_{CL}=\{k\mathbf{1,r}\}$ or Rogue Links

A non linear or **Rogue** $\mathbf{A}_{CL}=\{k\mathbf{1,r}\}$ cannot be manipulated so that the centers of its circles all lie on a straight line. Figure 6 is a Rogue link, 6r, consisting of 6 circles and is shown as a symmetrical plane projection. It can be designated as a member of $\mathbf{A}_{CL}=\{k\mathbf{1,r}\}$. This appears to be the smallest possible Rogue link and always has the centers of its circles in a non linear arrangement in flat projection no matter how the circles are manipulated.

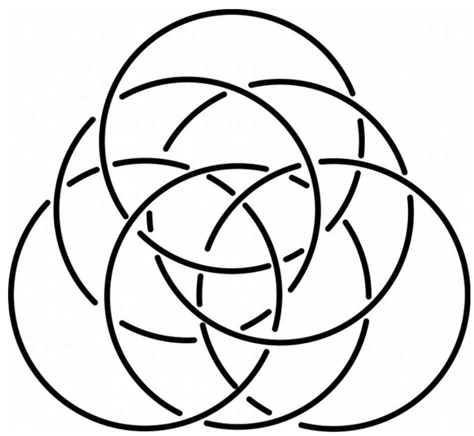


Figure 6

It is possible to arrange n integers in n! ways. Every one of these is a valid linear code for a valid linear linkage. However as n increases many of these codes represent the same linkage in in different fold arrangements. From the above discussion about linear linkages marked with binary powers we know that any linear prime can have up to n^2 unique binary sum arrangements with 2^(n-1) - n^2 possible non linear arrangements. For 5 prime this gives 16 - 25. A negative won't work. The reason it is negative is that many of the sums are identical but just represent a different linear fold arrangement. Less than 16 sums are possible for 5 prime. To find 6r in Figure 6 above, 5p was marked with binary powers. The marked 5p was manipulated into producing different binary sums to see if a linear arrangement was possible for each sum. One of the sums prevented a linear arrangement. That arrangement was then linked with a sixth circle thus locking the non linear in property and producing 6r above. A twist calculation for Rogue primes is to arrange the circles in a flat projection(possible for 6p) and then record the +1 or -1 twist of each pair of circles. At present no method for producing a construction code for nr has been worked out.

To make 7r it you can add a link to 6r when it is in the symmetrical arrangement in Figure 6. Alternatively you could use the same method on 6p to find 7r, as was used on 5p to find 6r above. An open question is to find an example of the smallest number of circles required to make a lock link . None of the circles of a lock link can be folded or laid over. A lock link should allow d/D to become smaller and smaller and stay locked.

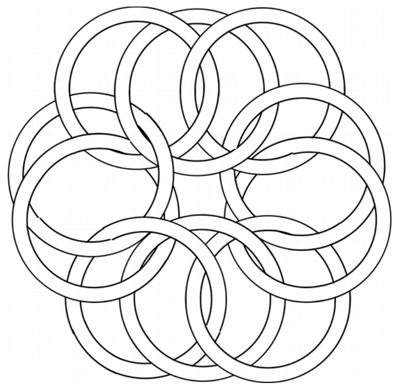


Figure 7

Figure 7 is an example of a k=6, q=4, n=10 linkage. It is a regular linkage $\mathbf{A_{CL}}$={k**6,q4,n10**}of circles where each circle has the same k and q. This linkage can fold inside out if the d/D ratio allows. Hold k constant or hold q constant as n changes to investigate properties of regular linkages.

Self organizing of close links

Close links always gather together in separate twisted groups. As the circles are made thicker this effect becomes more pronounced. The reason has to do with geometry. Close circles in a group occupy the least volume when they are close to each other in code order 1,2,3,4,…,n. They tend to exclude links that twist the opposite way. This is a type self organization in $\mathbf{A_{CL}}$={k**1,L**} architectures.

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