Reprinted with permission of **NCTM** magazine ***Mathematics Teacher***, October 1968, pp 571-574.
This article proves at least conceptually by a geometric topology that space can be twisted so much that it can surround a space forced to be untwisted. This is one of the research results of the Electrihedron-Atomihedron you can find much more research here:
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Some fascinating ideas, using a type of thinking somewhat similar to that required in modern polymer chemistry

CAN SPACE BE OVERTWISTED?

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AT FIRST sight the question "Can space be overtwisted?" seems absurd, if not ridiculous. In this article it will be shown that, given the right tools, space can be overtwisted under the meaning that these tools allow. All mathematical systems fit into a set of rules, and their meaning, consequently, does not go beyond certain boundaries unless the rules are expanded indefinitely. So if space can be overtwisted, it is with the restriction that you do it within a specific mathematical system.

We begin by taking a chain of geometrical links. Each link must be exactly alike. The link is the simplest geometrical solid, but it is not regular. It is a tetragonal disphenoid with four dihedral angles equal to 60° and two dihedral angles equal to 90°. Figure 1 shows a pattern for constructing such a link. The chain is made by taking some good adhesive tape and connecting the edges at a dihedral angle of 90°. We will not discuss all the possible rings or the proof for maximum twist in any ring. The proof does not seem to be necessary to derive the result we seek.

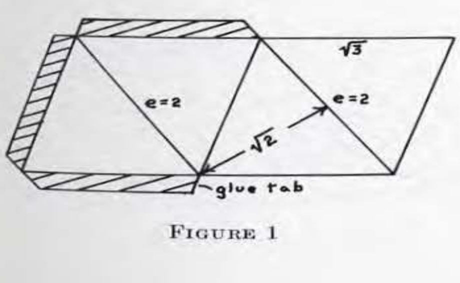

FIGURE 1

Certain writers on linkage mechanisms disregard the fact that chains of geometrical links can be given a twist. They refer to the idea that the same chain could have been constructed in such a way that complete 360° rotation is possible at all hinge joints. Hence they assume that twist is only degenerately important (1). However, the flexagons discussed by Martin Gardner in his monthly column in *Scientific American* require this twist to operate as they do (2). Twist is also very important in organic molecules, which cannot always be connected so that a 360° rotation can always occur at a rotatable joint. It shall be assumed that twist is mathematically important in these rings and especially in the result we seek.

It was found by trial and error that the maximum twist that could be gotten between the ends of a ring of $8n$ links was $t = (n/8) - 1$. Rings must be a multiple of 8 to have flexations that occur symmetrically. By "flex" we mean that the ring can, through a series of rotations of sets of links, be turned inside out any number of times without going backwards (3). Why must these rings be a multiple of 8 links for this symmetrical flexing property? This is because a ring of 8 links is the smallest possible. All larger rings consist of right-angle connections of the smallest ring. If you see it symmetrically, a maximum twist restricts additional links to be in positions similar to those occupied by the links in a ring of 8. Hence, applying symmetry, an equal number of links must appear across a point or line of symmetry, and this means that rings must always be a multiple of 8 to flex symmetrically.

The 16-link chain was twisted as hard as possible and then connected into a ring. It was then experimented with until a way was discovered to flex it symmetrically. Only one such way seems to exist for the 16-ring. Anyone who makes and attempts to flex these rings must be cautioned that they are not held in formation rigidly and will flop about many ways. One must learn to hold them in the position(s) that gives the symmetrical flexing pattern. This is not hard, but it makes for an extremely difficult time in discovering what the correct set of positions is in rings of 24 or more links.

A ring of more than 32 links is inflexible if given a maximum twist. This is a result of the fact that according to the formula for maximum twist, above, the twist per link increases and approaches $\frac{1}{8}$ as a limit as links are added. For the reason that flexing does not tell us whether space can be overtwisted we shall skip the flexing maneuvers.

Since, excluding the 8-ring, only three flexible rings are possible that flex symmetrically and have a maximum twist, one of these must contain maximum symmetry. That is, the flexible ring of the most links will probably contain too much twist per link and the ring of least links contain too little twist per link for the best symmetrical balance. Intuitively, a ring of 24 links should be the most symmetrical. The ring of 24 links is found to require 6 flexing cycles (a cycle is the amount a ring must be flexed before its positions begin to repeat) before positions of links appear again in the same place in each flex position of the ring. All other rings require as many cycles as there are links for this occurrence. The 24-link ring is also found to have a position where all links have a face in contact with a face of the connected link at each connection. This is the closed-angle position. An open angle is always present in all other maximum twist rings. The open angle is a place where two connected links have a 90° angle between their faces on both sides of their connecting hinge. Figure 2 shows the 24-link ring in the closed-angle position. This position now becomes important.

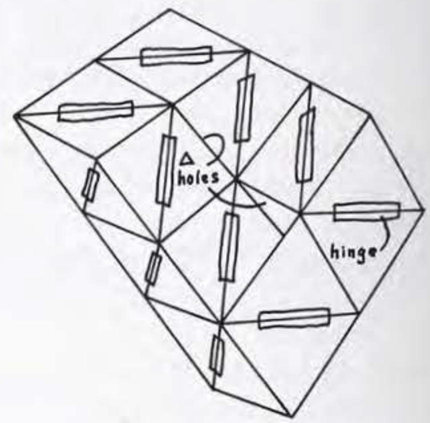

FIGURE 2

Figures 3 and 4 show how to take a rod that has an equilateral triangle for a cross section and cut it into congruent pieces that can then be glued into the shape of the closed-angle position of the 24-link ring. If the reader makes any of these (and they are fascinating to put together), he can have a lumber dealer saw them out of wood—in rods, and then sectioned as shown. The rods can be glued into any of

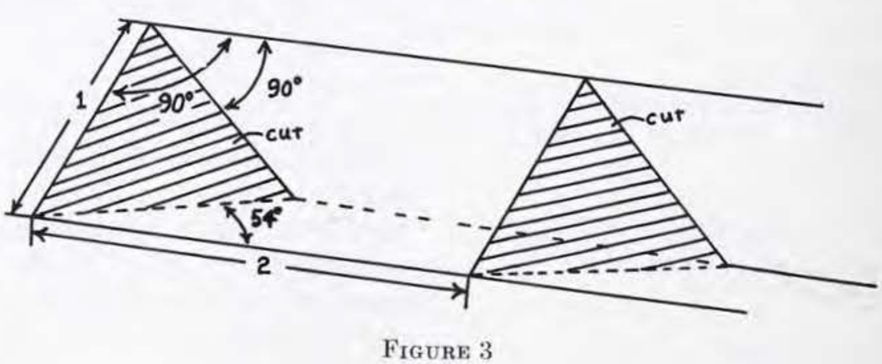

FIGURE 3

the rings of rings to be discussed below or any larger ones that one wishes to experiment with.

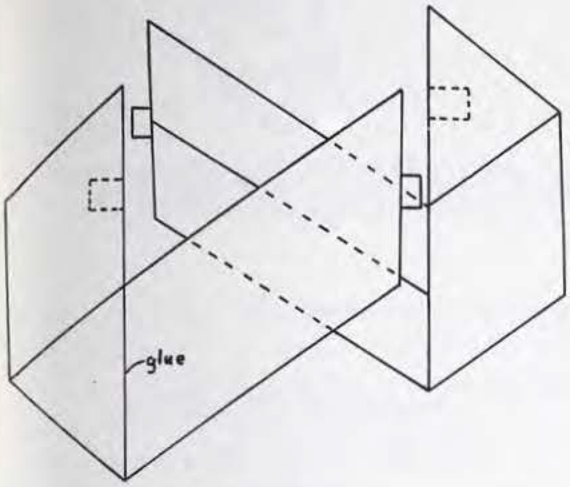

FIGURE 4

Figure 2 shows that the 24-ring has two symmetrically opposed triangular holes. Geometrically, these holes are equilateral triangles exactly equal in cross section to the rods the ring is made of. This is because of the 60° dihedral angles in each link. Rings that are twisted in the same sense can be linked together by placing a rod part of one ring in the hole part of another. Figure 5 shows two rings linked together symmetrically in this manner. We now find that there are two unsymmetrical ways that this linkage can be effected. One is geometrically different than the other. In fact, one way represents a rotation once away from the symmetrical linkage, and the other represents two rotations away from the symmetrical linkage.

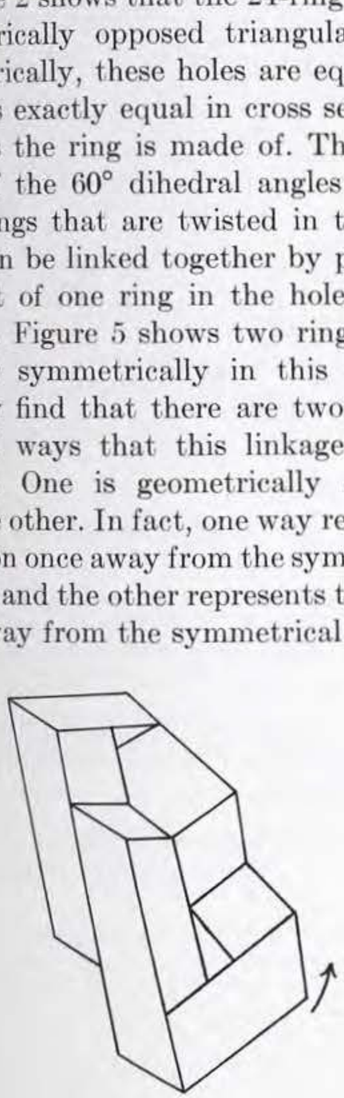

FIGURE 5

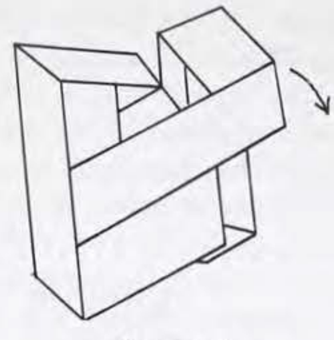

FIGURE 6

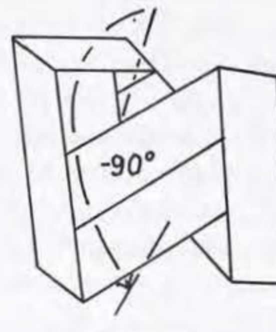

FIGURE 7

This is shown by the sequence of Figures 5 to 7. Two rotations, each in a different plane, describe something in three dimensions sometimes called a twist. In this case it is a twist of 90°. Hence if we make a ring of the solid links it will have a maximum twist if all connections are like those of Figure 7.

Figure 8 shows the completed ring we seek. It is composed of eight 24-link rings. Since each connection represents a 90° twist (see arrows in Fig. 7), the total twist is 720° or two. If the reader is good at geometrical visualizing, he can see that inside the 8-ring exists an unfilled space or void. Careful analysis shows that this void is a rhombic dodecahedron. A ring of 24 of the above tetragonal links can form this rhombic dodecahedron. If the reader has a chain of 24 links, he can find by experiment that it is impossible for such a ring to have a net twist between its ends. The rhombic dodecahedron is symmetric about three orthogonal axes, as seen in Figure 9. Also, it can be seen that one set of alternate angles all open outward as indicated in Figure 9. This forces the ring to bend upon itself, excluding any possible twist unless a knot is tied. Since it is impossible for the ring to form a knot, it must always have a zero twist.

Space is overtwisted in the 8-ring if it can be shown that less twist in a ring of 24 ring links would result in rings that allow the space inside and surrounding them to be twisted in the same sense and at least the same amount to create a reasonable fit. It is also required that any ring that has as much twist per link as the 8-ring overtwists space. Four smaller rings are possible. They are ones of 2, 6, 6, and 6 links, and all can fill space by replication or in combination with one another. Each of these rings has less twist per link than the 8-ring. Assuming that the 8-ring has a maximum twist and that the 24-ring it covers is untwisted by this maximum twist, it can be said that space is overtwisted by the 8-ring.

Analysis of Figure 8 shows that the 8-ring can fill space by replication by stacking similar to rhombic dodecahedra.

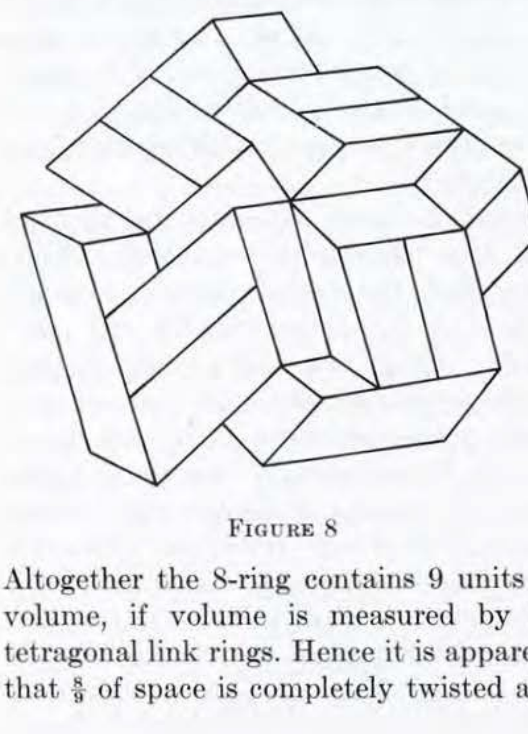

FIGURE 8

Altogether the 8-ring contains 9 units of volume, if volume is measured by 24 tetragonal link rings. Hence it is apparent that $\frac{8}{9}$ of space is completely twisted and

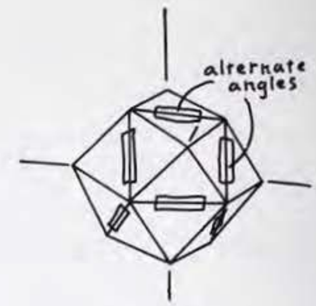

FIGURE 9

$\frac{1}{9}$ of space is completely nullified of twist by the 8-ring. The $\frac{1}{9}$ is completely contained and forced as a result of the maximum twist in the 8-ring. Therefore it can only be concluded that a part of space, at least $\frac{1}{9}$, is overtwisted by the 8-ring.

Discussion

If space can be overtwisted in the above manner, what does it mean? Does any reader know of any similar result? Does it mean that Euclidean space can be seen as curved? If Euclidean space can be curved, in what sense is it curved? Could this mean that ordinary perceptual space is antisymmetric? Does it mean that all mathematics is antisymmetric? Or is the result a complete fallacy? Perhaps it belongs to a realm of things that have no real meaning other than to a mathematician.

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