Reprinted with Permission of **Journal of Recreational Mathematics, JRM, V 4 No. 3,** 1971 pp-199-200. *(This brought two solutions both answered in the negative, which to my eternal hindsight now seems obvious. One response published in JRM V 11(4) 1978-79 by Dr. B.L Schwartz, see below. In 2018 I used this binary cube idea to put color dice dotted sticker squares 2x2x2 Rubik Cubes so that black and white parts are always the same, sold some.)*

An N-Dimensional Binary Coloring Problem

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It will take a four dimensional brain to solve this problem. No solution is known to the author. The problem concerns the binary coloring of n-dimensional binary solids. By an n-dimensional binary solid is meant a solid with edge equal to two units, all edges being perpendicular as in the square and cube. In two dimensions we are talking about a square with an area of four units, hence with an edge of two units. In three dimensions it would be a cube with a volume of eight units and edges of two units. In four dimensions it would be a hypercube with 16 unit cells. From this the reader can see what is meant by an n-dimensional binary solid.

There are only two ways to color the cells of a 2 x 2 square so that half of the cells are black and the other half white (excluding rotations and reflections). Both of these are symmetrical. That is, the pattern formed by the white squares is identical to the pattern formed by the black squares. There are six ways to color the cells of a 2 x 2 x 2 cube so that half of the cells are black and the other half white. In every one of these the black half is identical to the white half as the reader can see in the accompanying Figure.

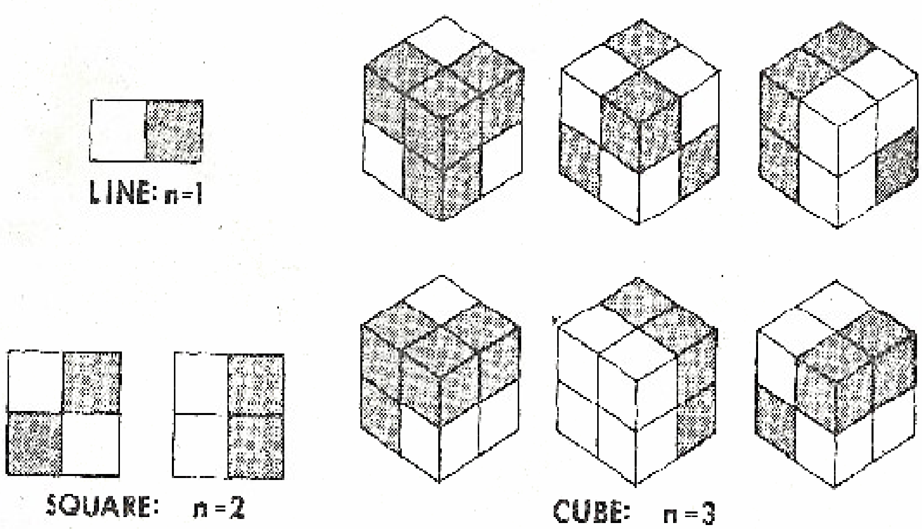

FIGURE 1.

The problem is therefore: Are all the binary colorings of an n-dimensional binary solid symmetrical? That is, is the pattern formed by the cluster of black cells identical to that formed by the white cells, no matter how they are colored half and half?

The problem may be easy to solve, but considering the rapid increase in the number of patterns as n increases it is more likely a difficult problem. A proof, if the symmetrical proposition is true, would require the consideration of a great number of symmetries as n increases. For instance, a cube has 13 elements of symmetry which crystallographers increase to 23 by considering planes and a center, and so on. This author has not been able to go beyond the cube. Therefore, it may well be that the proposition is untrue, but it could be true and it would be a pleasant exercise to produce the patterns that comprise the binary colorings of the binary hypercube.

(Here is the last of the the two page answer by Dr. Schwartz ***Black and White Vertices of a Hypercube*** *JRM V 11(4) 1978-79 where the black dots show the black colored 1x1x1 cubes)*

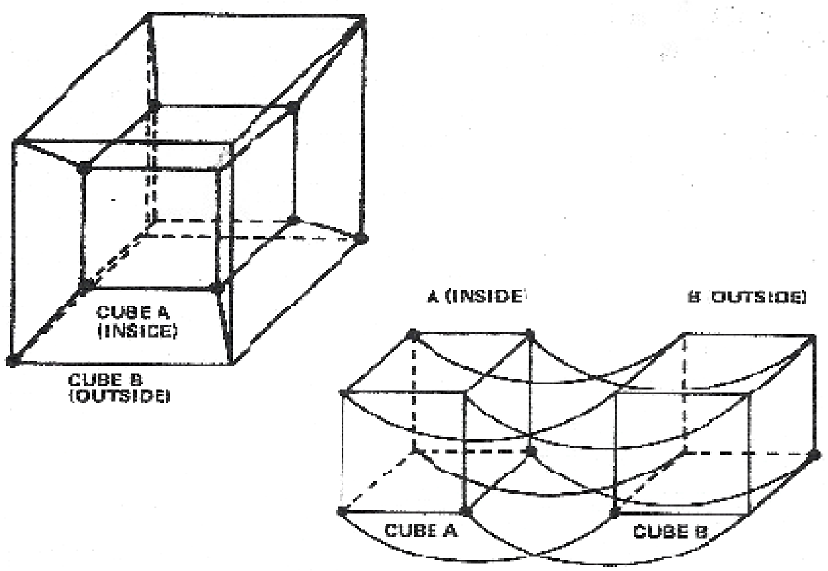

Figure 2. Noncongruent Black and White sets in the hypercube.

would have to be one with two Black vertices, and the only two Black vertices in Cube B are not on a common edge.

Since the White set includes an all-White square and the Black set does not include any all-Black squares, the two sets cannot be congruent.

REFERENCE

1. D. A. Engle, An N-Dimensional Binary Coloring Problem, *JRM*, 4:3, July 1971.

About the Author

Dr. Schwartz holds degrees in engineering, mathematics, and operations research, and is currently in graduate school again in a health care program. He has been an industrial mathematician and analyst with many organizations over the past twenty years, and is currently with Analytic Services, a private nonprofit study firm in the Washington, D.C. area. His fields of application include transportation, weapons systems, health, communications, recreation, and manpower. He has several dozen publications in the journals of mathematics, operations research, and computer science. For many years he has been a reviewer and/or editorial board member for the Operations Research Society of America, Association for Computing Machinery, Mathematical Association of America (*Mathematics Magazine*), and since 1970 an associate editor of *JRM*, to which he is a prolific contributor of both articles and problems. He has recently taken on the job of editing a book series on Recreational Mathematics for Baywood Publishing Co. Vol. 1 is due out soon.